Prince Edward Island

CANADA

Education and Early Childhood Development
English Programs

Prince Edward Island Mathematics Curriculum

CURRICULUM

Mathematics

MAT611B

CANADA

2013
Prince Edward Island
Department of Education and
Early Childhood Development
250 Water Street, Suite 101
Summerside, Prince Edward Island
Canada, C1N 1B6
Tel: (902) 438-4130
Fax: (902) 438-4062
www.gov.pe.ca/eecd/

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  J. Blaine Bernard,
  Secondary Mathematics Specialist,
  Department of Education and
  Early Childhood Development

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  Kirk Cutcliffe,
  Colonel Gray Senior High School

  Alan McAlduff,
  Westisle Composite High School

  Rob MacDonald,
  Colonel Gray Senior High School

  Glenda McInnis,
  Montague Regional High School

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Background and Rationale

The development of an effective mathematics curriculum has encompassed a solid research base. Developers have examined the curriculum proposed throughout Canada and secured the latest research in the teaching of mathematics, and the result is a curriculum that should enable students to understand and use mathematics.

The Western and Northern Canadian Protocol (WNCP) *Common Curriculum Framework for Grades 10-12 Mathematics* (2008) has been adopted as the basis for a revised high school mathematics curriculum in Prince Edward Island. The *Common Curriculum Framework* was developed by the seven Canadian western and northern ministries of education (British Columbia, Alberta, Saskatchewan, Manitoba, Yukon Territory, Northwest Territories, and Nunavut) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and achievement indicators agreed upon by the seven jurisdictions. This document is based on both national and international research by the WNCP, and on the *Principles and Standards for School Mathematics* (2000), published by the National Council of Teachers of Mathematics (NCTM).

➢ Essential Graduation Learnings

Essential Graduation Learnings (EGLs) are statements describing the knowledge, skills, and attitudes expected of all students who graduate from high school. Achievement of the essential graduation learnings will prepare students to continue to learn throughout their lives. These learnings describe expectations not in terms of individual school subjects but in terms of knowledge, skills, and attitudes developed throughout the curriculum. They confirm that students need to make connections and develop abilities across subject boundaries if they are to be ready to meet the shifting and ongoing demands of life, work, and study today and in the future. Essential graduation learnings are cross curricular, and curriculum in all subject areas is focused to enable students to achieve these learnings. Essential graduation learnings serve as a framework for the curriculum development process.

Specifically, graduates from the public schools of Prince Edward Island will demonstrate knowledge, skills, and attitudes expressed as essential graduation learnings, and will be expected to

- respond with critical awareness to various forms of the arts, and be able to express themselves through the arts;
- assess social, cultural, economic, and environmental interdependence in a local and global context;
- use the listening, viewing, speaking, and writing modes of language(s), and mathematical and scientific concepts and symbols, to think, learn, and communicate effectively;
- continue to learn and to pursue an active, healthy lifestyle;
- use the strategies and processes needed to solve a wide variety of problems, including those requiring language, and mathematical and scientific concepts;
- use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems.

More specifically, curriculum outcome statements articulate what students are expected to know and be able to do in particular subject areas. Through the achievement of curriculum outcomes, students demonstrate the essential graduation learnings.

➢ Curriculum Focus

There is an emphasis in the Prince Edward Island mathematics curriculum on particular key concepts at each grade which will result in greater depth of understanding. There is also more emphasis on number sense and operations in the early grades to ensure students develop a solid foundation in numeracy. The intent of this document is to clearly communicate to all educational partners high expectations for students in mathematics education. Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge (NCTM, *Principles and Standards for School Mathematics*, 2000).

The main goals of mathematics education are to prepare students to

- use mathematics confidently to solve problems;
- communicate and reason mathematically;
- appreciate and value mathematics;
- make connections between mathematics and its applications;
- commit themselves to lifelong learning;
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will

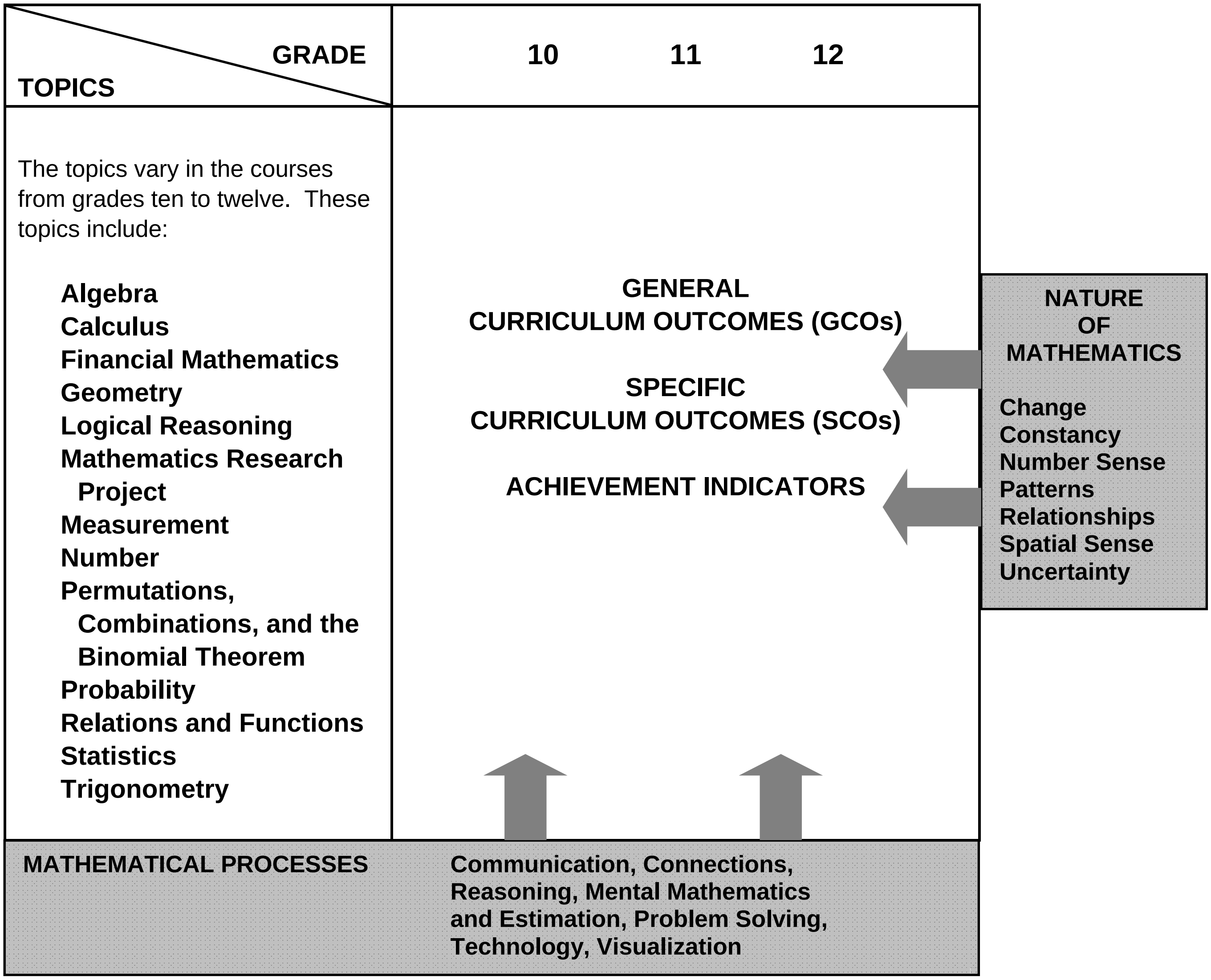
- gain understanding and appreciation of the contributions of mathematics as a science, philosophy, and art;
- exhibit a positive attitude toward mathematics;
- engage and persevere in mathematical tasks and projects;
- contribute to mathematical discussions;
- take risks in performing mathematical tasks;
- exhibit curiosity.

➢ Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the understanding of mathematical concepts by students and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in literacy, science, social studies, music, art, physical education, and other subject areas. Efforts should be made to make connections and use examples drawn from a variety of disciplines.

Conceptual Framework for 10-12 Mathematics

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



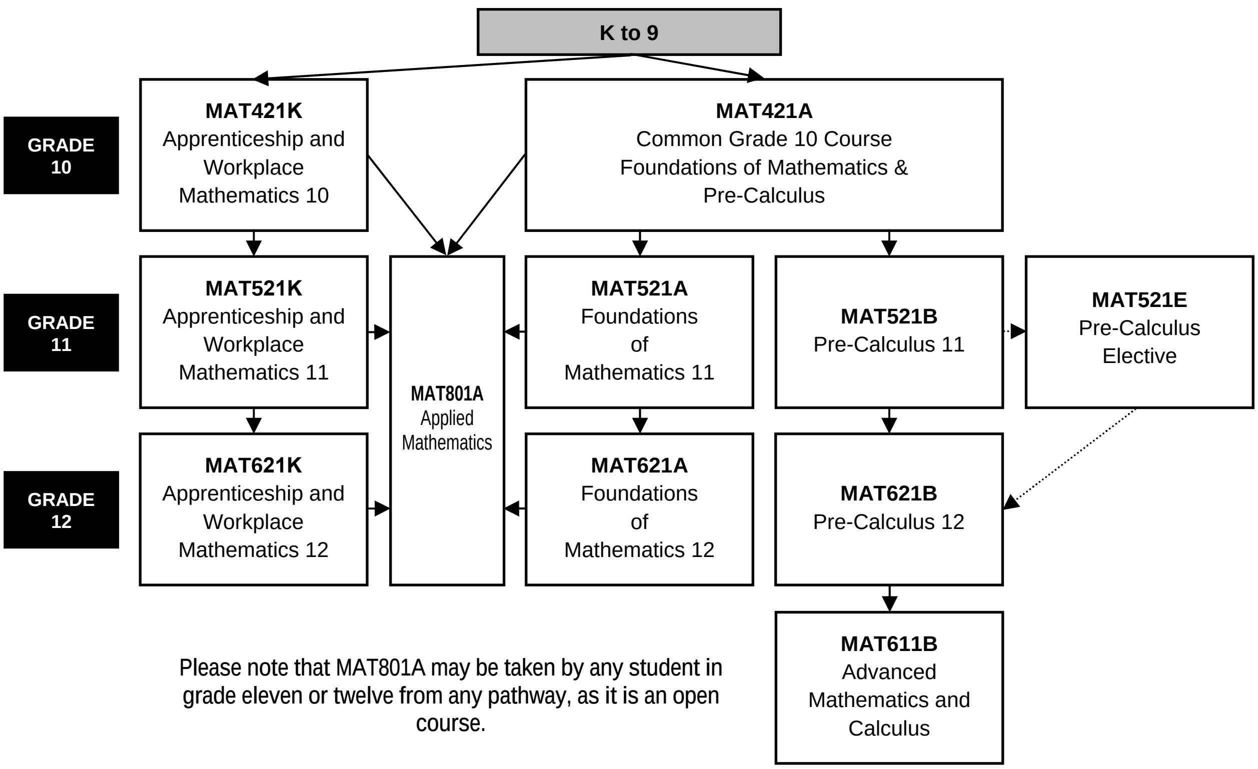
The mathematics curriculum describes the nature of mathematics, as well as the mathematical processes and the mathematical concepts to be addressed. This curriculum is arranged into a number of topics, as described above. These topics are not intended to be discrete units of instruction. The integration of outcomes across topics makes mathematical experiences meaningful. Students should make the connections among concepts both within and across topics. Consider the following when planning for instruction:

- Integration of the mathematical processes within each topic is expected.
- Decreasing emphasis on rote calculation, drill, and practice, and the size of numbers used in paper and pencil calculations makes more time available for concept development.

- Problem solving, reasoning, and connections are vital to increasing mathematical fluency, and must be integrated throughout the program.
- There is to be a balance among mental mathematics and estimation, paper and pencil exercises, and the use of technology, including calculators and computers. Concepts should be introduced using models and gradually developed from the concrete to the pictorial to the symbolic.

➢ Pathways and Topics

The Prince Edward Island 10-12 mathematics curriculum includes pathways with corresponding topics rather than strands, which are found in the Prince Edward Island K-9 mathematics curriculum. Three pathways are available: **Apprenticeship and Workplace Mathematics**, **Foundations of Mathematics**, and **Pre-Calculus**. A common grade ten course (Foundations of Mathematics and Pre-Calculus, Grade 10) is the starting point for the Foundations of Mathematics pathway and the Pre-Calculus pathway. Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings. These pathways are illustrated in the diagram below:



The goals of all three pathways are to provide the prerequisite knowledge, skills, understandings, and attitudes for specific post-secondary programs or direct entry into the work force. All three pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. Each pathway is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific post-secondary programs of study or for direct entry into the work force. When choosing a pathway, students should consider their interests, both current and future. Students, parents and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Apprenticeship and Workplace Mathematics

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into the majority of trades and for direct entry into the work force. Topics include algebra, geometry, measurement, number, statistics, and probability.

Foundations of Mathematics

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, measurement, algebra and number, logical reasoning, relations and functions, statistics, probability, and a mathematics research project.

Pre-Calculus

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Topics include algebra and number, measurement, relations and functions, trigonometry, combinatorics, and introductory calculus.

➢ Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics. The Prince Edward Island mathematics curriculum incorporates the following seven interrelated mathematical processes that are intended to permeate teaching and learning. These unifying concepts serve to link the content to methodology.

Students are expected to

- communicate in order to learn and express their understanding of mathematics; **[Communications: C]**
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines; **[Connections: CN]**
- demonstrate fluency with mental mathematics and estimation; **[Mental Mathematics and Estimation: ME]**
- develop and apply new mathematical knowledge through problem solving; **[Problem Solving: PS]**
- develop mathematical reasoning; **[Reasoning: R]**
- select and use technologies as tools for learning and solving problems; **[Technology: T]**
- develop visualization skills to assist in processing information, making connections, and solving problems. **[Visualization: V]**

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology. Communication can help students make connections among concrete, pictorial, symbolic, verbal, written, and mental representations of mathematical ideas.

Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

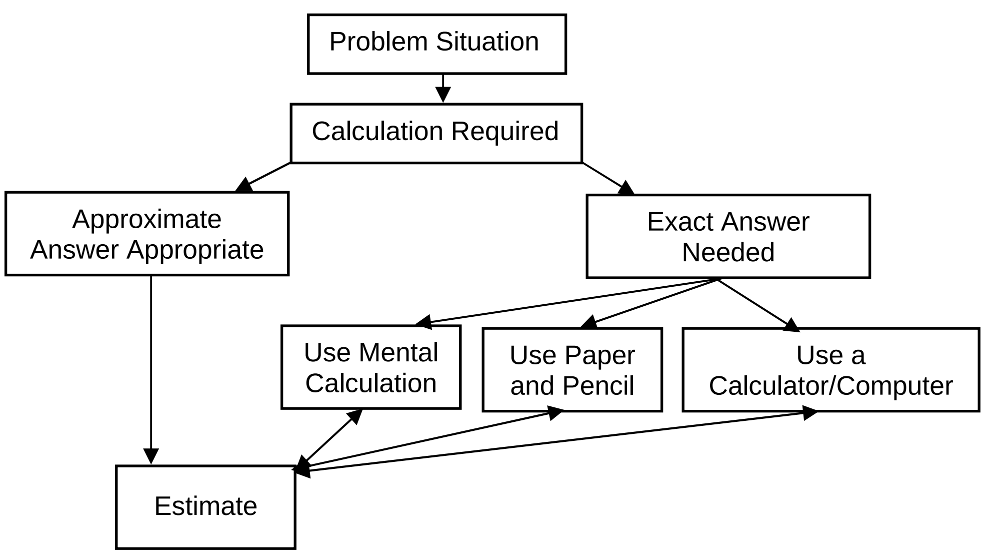
For instance, opportunities should be created frequently to link mathematics and career opportunities. Students need to become aware of the importance of mathematics and the need for mathematics in many career paths. This realization will help maximize the number of students who strive to develop and maintain the mathematical abilities required for success in further areas of study.

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It involves calculation without the use of external memory aids. Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy, and flexibility. Even more important than performing computational procedures or using calculators is the greater facility that students need - more than ever before - with estimation and mental mathematics (National Council of Teachers of Mathematics, May 2005). Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving” (Rubenstein, 2001). Mental mathematics “provides a cornerstone for all estimation processes offering a variety of alternate algorithms and non-standard techniques for finding answers” (Hope, 1988).

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know when to estimate, what strategy to use, and how to use it. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process described below:



(NCTM)

Problem Solving [PS]

Learning through problem solving should be the focus of mathematics at all grade levels. When students encounter new situations and respond to questions of the type, “How would you. . . ?" or “How could you. . . ?” the problem-solving approach is being modelled. Students develop their own problem-solving strategies by being open to listening, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement.

Problem solving is also a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for and engage in finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident and cognitive mathematical risk takers.

Over time, numerous problem-solving strategies should be modelled for students, and students should be encouraged to employ various strategies in many problem-solving situations. While choices with respect to the timing of the introduction of any given strategy will vary, the following strategies should all become familiar to students:

- using estimation
- guessing and checking
- looking for a pattern
- making an organized list or table
- using a model
- working backwards
- using a formula
- using a graph, diagram, or flow chart
- solving a simpler problem
- using algebra.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. High-order questions challenge students to think and develop a sense of wonder about mathematics. Mathematical experiences in and out of the classroom provide opportunities for inductive and deductive reasoning. Inductive reasoning occurs when students explore and record results, analyse observations, make generalizations from patterns, and test these generalizations. Deductive reasoning occurs when students reach new conclusions based upon what is already known or assumed to be true.

Technology [T]

Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Calculators and computers can be used to

- explore and demonstrate mathematical relationships and patterns;
- organize and display data;
- extrapolate and interpolate;
- assist with calculation procedures as part of solving problems;
- decrease the time spent on computations when other mathematical learning is the focus;
- reinforce the learning of basic facts and test properties;
- develop personal procedures for mathematical operations;
- create geometric displays;
- simulate situations;

- develop number sense.

Technology contributes to a learning environment in which the growing curiosity of students can lead to rich mathematical discoveries at all grade levels. While technology can be used in K-3 to enrich learning, it is expected that students will meet all outcomes without the use of technology.

Visualization [V]

Visualization involves thinking in pictures and images, and the ability to perceive, transform, and recreate different aspects of the visual-spatial world. The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret, and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure and when to estimate, and knowledge of several estimation strategies (Shaw & Cliatt, 1989).

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations.

➢ The Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics which are woven throughout this document. These components include change, constancy, number sense, patterns, relationships, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4;
- an arithmetic sequence, with first term 4 and a common difference of 2; or
- a linear function with a discrete domain.

Constancy

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state, and symmetry (AAAS–Benchmarks, 1993, p. 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180^0.

- The theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations, or the angle sums of polygons.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy (*The Primary Program*, B.C., 2000, p. 146). A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Number sense develops when students connect numbers to real-life experiences, and use benchmarks and referents. This results in students who are computationally fluent, and flexible and intuitive with numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all topics in mathematics and it is important that connections are made among topics. Working with patterns enables students to make connections within and beyond mathematics. These skills contribute to students' interaction with and understanding of their environment. Patterns may be represented in concrete, visual, or symbolic form. Students should develop fluency in moving from one representation to another. Students must learn to recognize, extend, create, and use mathematical patterns. Patterns allow students to make predictions and justify their reasoning when solving routine and non-routine problems. Learning to work with patterns in the early grades helps develop students' algebraic thinking that is foundational for working with more abstract mathematics in higher grades.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, and concepts. The search for possible relationships involves the collecting and analysing of data, and describing relationships visually, symbolically, orally, or in written form.

Spatial Sense

Spatial sense involves visualization, mental imagery, and spatial reasoning. These skills are central to the understanding of mathematics. Spatial sense enables students to interpret representations of 2-D shapes and 3-D objects, and identify relationships to mathematical topics. Spatial sense is developed through a variety of experiences and interactions within the environment. The development of spatial sense enables students to solve problems involving 2-D shapes and 3-D objects.

Spatial sense offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations. Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to use dimensions and make predictions about the results of changing dimensions.

- Knowing the dimensions of an object enables students to communicate about the object and create representations.
- The volume of a rectangular solid can be calculated from given dimensions.
- Doubling the length of the side of a square increases the area by a factor of four.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately.

Contexts for Learning and Teaching

The Prince Edward Island mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice:

- Mathematics learning is an active and constructive process.
- Learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates.
- Learning is most likely to occur in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and that nurtures positive attitudes and sustained effort.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Young children develop a variety of mathematical ideas before they enter school. They make sense of their environment through observations and interactions at home and in the community. Their mathematics learning is embedded in everyday activities, such as playing, reading, storytelling, and helping around the home. Such activities can contribute to the development of number and spatial sense in children. Initial problem solving and reasoning skills are fostered when children are engaged in activities such as comparing quantities, searching for patterns, sorting objects, ordering objects, creating designs, building with blocks, and talking about these activities. Positive early experiences in mathematics are as critical to child development as are early literacy experiences.

Students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. The use of models and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students, and enhance the formation of sound, transferable, mathematical concepts. At all levels, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful discussions can provide essential links among concrete, pictorial, and symbolic representations of mathematics.

The learning environment should value and respect the experiences and ways of thinking of all students, so that learners are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. Learners must be encouraged that it is acceptable to solve problems in different ways and realize that solutions may vary.

➢ Homework

Homework is an essential component of the mathematics program, as it extends the opportunity for students to think mathematically and to reflect on ideas explored during class time. The provision of this additional time for reflection and practice plays a valuable role in helping students to consolidate their learning.

Traditionally, homework has meant completing ten to twenty drill and practice questions relating to the procedure taught in a given day. With the increased emphasis on problem solving, conceptual understanding, and mathematical reasoning, however, it is important that homework assignments change accordingly. More assignments involving problem solving, mathematical investigations, written explanations and reflections, and data collection should replace some of the basic practice exercises given in isolation. In fact, a good problem can sometimes accomplish more than many drill-oriented exercises on a topic.

As is the case in designing all types of homework, the needs of the students and the purpose of the assignment will dictate the nature of the questions included. Homework need not be limited to reinforcing learning; it provides an excellent opportunity to revisit topics explored previously and to introduce new topics before teaching them in the classroom. Homework provides an effective way to communicate with parents and provides parents an opportunity to be actively involved in their child's learning. By ensuring that assignments model classroom instruction and sometimes require parental input, a teacher can give a parent clearer understanding of the mathematics curriculum and of the child's progress in relationship to it. As Van de Walle (1994, p. 454) suggests, homework can serve as a parent's window to the classroom.

➢ Diversity in Student Needs

Every class has students at many different cognitive levels. Rather than choosing a certain level at which to teach, a teacher is responsible for tailoring instruction to reach as many of these students as possible. In general, this may be accomplished by assigning different tasks to different students or assigning the same open-ended task to most students. Sometimes it is appropriate for a teacher to group students by interest or ability, assigning them different tasks in order to best meet their needs. These groupings may last anywhere from minutes to semesters, but should be designed to help all students (whether strong, weak or average) to reach their highest potential. There are other times when an appropriately open-ended task can be valuable to a broad spectrum of students. For example, asking students to make up an equation for which the answer is 5 allows some students to make up very simple equations while others can design more complex ones. The different equations constructed can become the basis for a very rich lesson from which all students come away with a better understanding of what the solution to an equation really means.

➢ Gender and Cultural Equity

The mathematics curriculum and mathematics instruction must be designed to equally empower both male and female students, as well as members of all cultural backgrounds. Ultimately, this should mean not only that enrolments of students of both genders and various cultural backgrounds in public school mathematics courses should reflect numbers in society, but also that representative numbers of both genders and the various cultural backgrounds should move on to successful post-secondary studies and careers in mathematics and mathematics-related areas.

➢ Mathematics for EAL Learners

The Prince Edward Island mathematics curriculum is committed to the principle that learners of English as an additional language (EAL) should be full participants in all aspects of mathematics education. English deficiencies and cultural differences must not be barriers to full participation. All students should study a comprehensive mathematics curriculum with high-quality instruction and co-ordinated assessment.

The *Principles and Standards for School Mathematics* (NCTM, 2000) emphasizes communication "as an essential part of mathematics and mathematics education" (p.60). The *Standards* elaborate that all students, and EAL learners in particular, need to have opportunities and be given encouragement and support for speaking, writing, reading, and listening in mathematics classes. Such efforts have the potential to help EAL learners overcome barriers and will facilitate “communicating to learn mathematics and learning to communicate mathematically” (NCTM, p.60).

To this end,

- schools should provide EAL learners with support in their dominant language and English language while learning mathematics;
- teachers, counsellors, and other professionals should consider the English-language proficiency level of EAL learners as well as their prior course work in mathematics;
- the mathematics proficiency level of EAL learners should be solely based on their prior academic record and not on other factors;
- mathematics teaching, curriculum, and assessment strategies should be based on best practices and build on the prior knowledge and experiences of students and on their cultural heritage;
- the importance of mathematics and the nature of the mathematics program should be communicated with appropriate language support to both students and parents;
- to verify that barriers have been removed, educators should monitor enrolment and achievement data to determine whether EAL learners have gained access to, and are succeeding in, mathematics courses.

➢ Education for Sustainable Development

Education for sustainable development (ESD) involves incorporating the key themes of sustainable development - such as poverty alleviation, human rights, health, environmental protection, and climate change - into the education system. ESD is a complex and evolving concept and requires learning about these key themes from a social, cultural, environmental, and economic perspective, and exploring how those factors are interrelated and interdependent.

With this in mind, it is important that all teachers, including mathematics teachers, attempt to incorporate these key themes in their subject areas. One tool that can be used is the searchable on-line database *Resources for Rethinking*, found at **http://r4r.ca/en**. It provides teachers with access to materials that integrate ecological, social, and economic spheres through active, relevant, interdisciplinary learning.

➢ Inquiry-Based Learning and Project Based Learning

Inquiry-based learning (IBL) allows students to explore, investigate, and construct new meaning from prior knowledge and from new information that is retrieved from other sources. It is not linear in nature, but promotes a continual looping back and forth throughout the process as students gather and process new information, redirect their inquiries, and continue through the process. Mathematical inquiry will require students to practise and refine their critical and creative-thinking skills. The terms *inquiry* and *research* are often used interchangeably within an educational context. While research often becomes the end-result of an inquiry process, it is the process itself that should be emphasized within an educational context. More information regarding the development of a mathematics research project is included in the appendix at the end of this document.

Assessment and Evaluation

Assessment and evaluation are essential components of teaching and learning in mathematics. The basic principles of assessment and evaluation are as follows:

- Effective assessment and evaluation are essential to improving student learning.
- Effective assessment and evaluation are aligned with the curriculum outcomes.
- A variety of tasks in an appropriate balance gives students multiple opportunities to demonstrate their knowledge and skills.
- Effective evaluation requires multiple sources of assessment information to inform judgments and decisions about the quality of student learning.
- Meaningful assessment data can demonstrate student understanding of mathematical ideas, student proficiency in mathematical procedures, and student beliefs and attitudes about mathematics.

Without effective assessment and evaluation it is impossible to know whether students have learned, teaching has been effective, or how best to address student learning needs. The quality of assessment and evaluation in the educational process has a profound and well-established link to student performance. Research consistently shows that regular monitoring and feedback are essential to improving student learning. What is assessed and evaluated, how it is assessed and evaluated, and how results are communicated send clear messages to students and others.

➢ Assessment

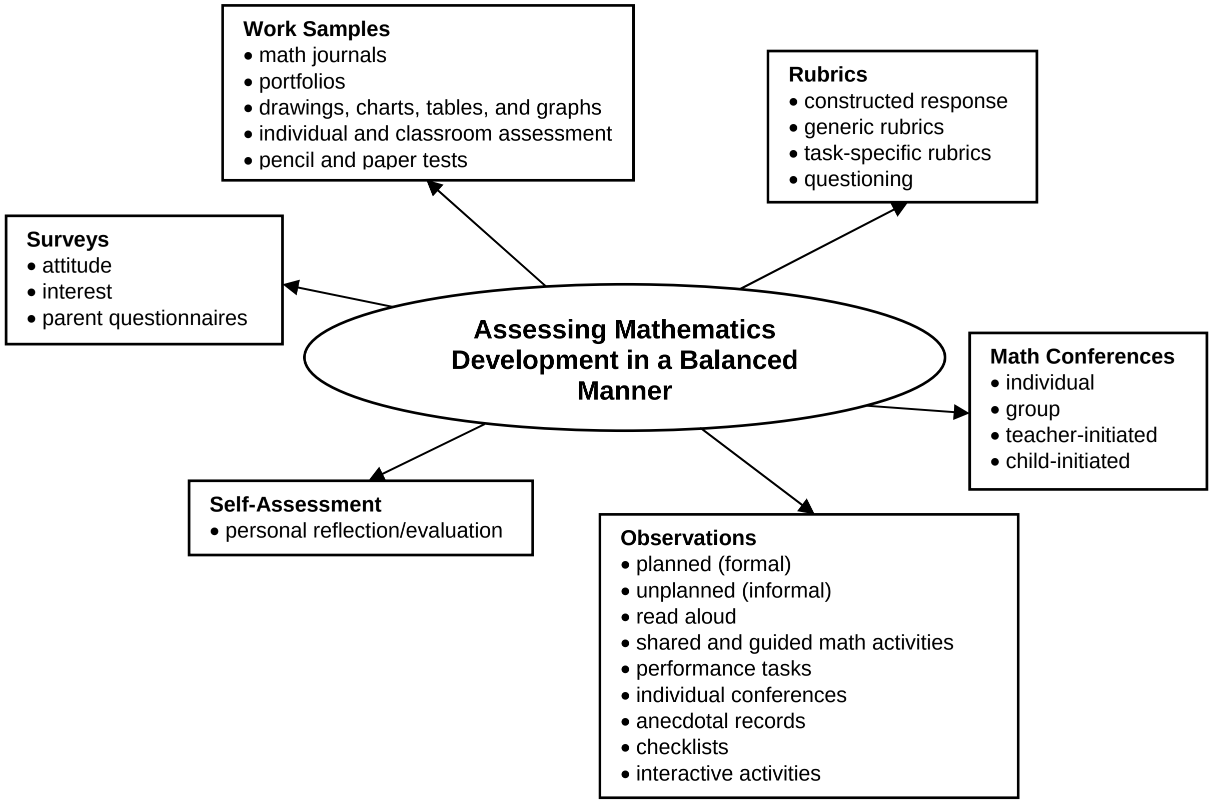
Assessment is the systematic process of gathering information on student learning. To determine how well students are learning, assessment strategies have to be designed to systematically gather information on the achievement of the curriculum outcomes. Teacher-developed assessments have a wide variety of uses, such as

- providing feedback to improve student learning;
- determining if curriculum outcomes have been achieved;
- certifying that students have achieved certain levels of performance;
- setting goals for future student learning;
- communicating with parents about their children's learning;
- providing information to teachers on the effectiveness of their teaching, the program, and the learning environment;
- meeting the needs of guidance and administration.

A broad assessment plan for mathematics ensures a balanced approach to summarizing and reporting. It should consider evidence from a variety of sources, including

- formal and informal observations
- work samples
- anecdotal records
- conferences
- teacher-made and other tests
- portfolios
- learning journals
- questioning
- performance assessment
- peer- and self-assessment.

This balanced approach for assessing mathematics development is illustrated in the diagram on the next page.



There are three interrelated purposes for classroom assessment: assessment *as* learning, assessment *for* learning, and assessment *of* learning. Characteristics of each type of assessment are highlighted below.

Assessment *as* learning is used

- to engage students in their own learning and self-assessment;
- to help students understand what is important in the mathematical concepts and particular tasks they encounter;
- to develop effective habits of metacognition and self-coaching;
- to help students understand themselves as learners - *how* they learn as well as *what* they learn - and to provide strategies for reflecting on and adjusting their learning.

Assessment *for* learning is used

- to gather and use ongoing information in relation to curriculum outcomes in order to adjust instruction and determine next steps for individual learners and groups;
- to identify students who are at risk, and to develop insight into particular needs in order to differentiate learning and provide the scaffolding needed;
- to provide feedback to students about how they are doing and how they might improve;
- to provide feedback to other professionals and to parents about how to support students' learning.

Assessment *of* learning is used

- to determine the level of proficiency that a student has demonstrated in terms of the designated learning outcomes for a unit or group of units;
- to facilitate reporting;

- to provide the basis for sound decision-making about next steps in a student's learning.

➢ Evaluation

Evaluation is the process of analysing, reflecting upon, and summarizing assessment information, and making judgments or decisions based upon the information gathered. Evaluation involves teachers and others in analysing and reflecting upon information about student learning gathered in a variety of ways.

This process requires

- developing clear criteria and guidelines for assigning marks or grades to student work;
- synthesizing information from multiple sources;
- weighing and balancing all available information;
- using a high level of professional judgment in making decisions based upon that information.

➢ Reporting

Reporting on student learning should focus on the extent to which students have achieved the curriculum outcomes. Reporting involves communicating the summary and interpretation of information about student learning to various audiences who require it. Teachers have a special responsibility to explain accurately what progress students have made in their learning and to respond to parent and student inquiries about learning. Narrative reports on progress and achievement can provide information on student learning which letter or number grades alone cannot. Such reports might, for example, suggest ways in which students can improve their learning and identify ways in which teachers and parents can best provide support. Effective communication with parents regarding their children's progress is essential in fostering successful home-school partnerships. The report card is one means of reporting individual student progress. Other means include the use of conferences, notes, phone calls, and electronic methods.

➢ Guiding Principles

In order to provide accurate, useful information about the achievement and instructional needs of students, certain guiding principles for the development, administration, and use of assessments must be followed. The document *Principles for Fair Student Assessment Practices for Education in Canada* (1993) articulates five fundamental assessment principles, as follows:

- Assessment methods should be appropriate for and compatible with the purpose and context of the assessment.
- Students should be provided with sufficient opportunity to demonstrate the knowledge, skills, attitudes, or behaviours being assessed.
- Procedures for judging or scoring student performance should be appropriate for the assessment method used and be consistently applied and monitored.
- Procedures for summarizing and interpreting assessment results should yield accurate and informative representations of a student's performance in relation to the curriculum outcomes for the reporting period.
- Assessment reports should be clear, accurate, and of practical value to the audience for whom they are intended.

These principles highlight the need for assessment which ensures that

- the best interests of the student are paramount;

- assessment informs teaching and promotes learning;
- assessment is an integral and ongoing part of the learning process and is clearly related to the curriculum outcomes;
- assessment is fair and equitable to all students and involves multiple sources of information.

While assessments may be used for different purposes and audiences, all assessments must give each student optimal opportunity to demonstrate what he or she knows and can do.

Structure and Design of the Curriculum Guide

The learning outcomes in the Prince Edward Island high school mathematics curriculum are organized into a number of topics across the grades from ten to twelve. Each topic has associated with it a general curriculum outcome (GCO). They are overarching statements about what students are expected to learn in each topic from grades ten to twelve.

Topic	General Curriculum Outcome (GCO)
Algebra (A)	Develop algebraic reasoning.
Algebra and Number (AN)	Develop algebraic reasoning and number sense.
Calculus (C)	Develop introductory calculus reasoning.
Financial Mathematics (FM)	Develop number sense in financial applications.
Geometry (G)	Develop spatial sense.
Logical Reasoning (LR)	Develop logical reasoning.
Mathematics Research Project (MRP)	Develop an appreciation of the role of mathematics in society.
Measurement (M)	Develop spatial sense and proportional reasoning. **(*Foundations of Mathematics and Pre-Calculus*)**
	Develop spatial sense through direct and indirect measurement. **(*Apprenticeship and Workplace Mathematics*)**
Number (N)	Develop number sense and critical thinking skills.
Permutations, Combinations and Binomial Theorem (PC)	Develop algebraic and numeric reasoning that involves combinatorics.
Probability (P)	Develop critical thinking skills related to uncertainty.
Relations and Functions (RF)	Develop algebraic and graphical reasoning through the study of relations.
Statistics (S)	Develop statistical reasoning.
Trigonometry (T)	Develop trigonometric reasoning.

Each general curriculum outcome is then subdivided into a number of specific curriculum outcomes (SCOs). Specific curriculum outcomes are statements that identify the specific skills, understandings, and knowledge students are required to attain by the end of a given grade.

Finally, each specific curriculum outcome has a list of achievement indicators that are used to determine whether students have met the corresponding specific curriculum outcome.

In this curriculum guide, each specific curriculum outcome (SCO) is presented in a two-page format, and includes the following information:

- its corresponding topic and general curriculum outcome;
- the specific curriculum outcome, with a list of achievement indicators;
- a list of the sections in *Calculus: Graphical, Numerical, Algebraic* which address the SCO, with specific achievement indicators highlighted in brackets;
- an elaboration for the SCO.

In the second half of this document, a curriculum guide supplement is presented which follows the primary resource, *Calculus: Graphical, Numerical, Algebraic*, with additional references to the secondary resource, *Calculus* [7E]. As well, an appendix is included which outlines the steps to follow in the development of an effective mathematics research project.

CALCULUS

SPECIFIC CURRICULUM OUTCOMES

C1 – Solve problems involving limits.

C2 – Solve problems involving asymptotic and end behaviour.

C3 – Solve problems involving continuity.

C4 – Explore the concept of the derivative as the instantaneous rate of change.

C5 – Determine the derivative of a function by applying the definition of derivative.

C6 – Apply derivative rules to determine the derivative of a function, including:

- **Constant Rule;**
- **Power Rule;**
- **Constant Multiple Rule;**
- **Sum Rule;**
- **Difference Rule;**
- **Product Rule;**
- **Quotient Rule.**

C7 – Use calculus techniques to solve problems involving rates of change, including motion problems involving position, velocity, and acceleration.

C8 – Find derivatives of trigonometric functions.

C9 – Apply the Chain Rule to determine the derivative of a function.

C10 – Solve problems involving inverse trigonometric functions.

C11 – Find limits and derivatives of exponential and logarithmic functions.

C12 – Use calculus techniques to sketch the graph of a function.

C13 – Use calculus techniques to solve optimization problems.

C14 – Use linearization and Newton’s Method to solve problems.

C15 – Solve problems involving related rates.

C16 – Determine the indefinite integral of a function.

C17 – Determine the definite integral of a function.

C18 – Solve problems that involve the application of the integral of a function.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C1 – Solve problems involving limits.** [C, CN, ME, PS, R, T, V]

Students who have achieved this outcome should be able to:

A. Calculate the average speed of an object over a given time interval.

B. Determine the instantaneous speed of an object at a given time.

C. Explore the concept of limit, using informal methods.

D. Apply the properties of limits to solve limit problems.

E. Determine the value of the limit of a function as the variable approaches a real number by using

- a table of values;
- a graph;
- algebraic manipulation;
- substitution.

F. Establish each of the following trigonometric limits, using informal methods:

- $\lim_{x \to 0} \sin x = 0$;
- $\lim_{x \to 0} \cos x = 1$;
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$;
- $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

G. Determine when the value of a limit of a function does not exist.

H. Determine the value of the left-handed and the right-handed limits of a function at a given point, from its graph or by using algebraic methods.

I. Determine the value of the limit of a piecewise function as the variable approaches a real number.

J. Use the Sandwich Theorem to find certain limits indirectly.

K. Establish the limit $\lim_{x \to \infty} \frac{1}{x} = 0$, using informal methods.

L. Determine the value of the limit of a function as the variable approaches positive infinity and negative infinity.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

2.1 (A B C D E F G H I J)

2.2 (K L)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: C1 – Solve problems involving limits. [C, CN, ME, PS, R, T, V]

Elaboration

The average speed of a moving body, defined by $y = f(t)$, during an interval of time from *t* to $t+h$, is found by dividing the distance covered, Δy, by the elapsed time, Δt, or

$$\frac{\Delta y}{\Delta t} = \frac{f(t+h) - f(t)}{h}$$

This expression can be used to find the instantaneous speed at a given time, *t*, by letting *h* approach zero. This is an intuitive way of introducing the concept of limit.

For reference, the concept of limit is formally defined in the textbook, however the concept should be introduced using informal methods in this course. Using graphs and technology, such as a graphing calculator, can help give students an appreciation of the concept of limit.

Students should be comfortable working with the properties of limits. If *L*, *M*, *c*, and *k* are real numbers, and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then the following limit rules apply.

NAME	RULE
Sum Rule	$\lim_{x \to c} \left[f(x) + g(x) \right] = L + M$
Difference Rule	$\lim_{x \to c} \left[f(x) - g(x) \right] = L - M$
Product Rule	$\lim_{x \to c} \left[f(x) \bullet g(x) \right] = L \bullet M$
Constant Multiple Rule	$\lim_{x \to c} \left[k \bullet f(x) \right] = k \bullet L$
Quotient Rule	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$
Power Rule	$\lim_{x \to c} \left[f(x) \right]^k = L^k$

When working with limits involving the indeterminate form $\frac{0}{0}$, students will have to algebraically manipulate the expression in order to evaluate the limit. There are a number of methods that can be used depending on the problem, including simplifying, factoring, rationalizing, and rewriting trigonometric expressions.

Another method that can be used to evaluate certain limits is by using the Sandwich Theorem. If we can establish that the limit that we are looking for is between two other limits that have the same value, then the third limit must also have that same value.

After having established the limit $\lim_{x \to \infty} \frac{1}{x} = 0$ using informal methods, students can use it to evaluate the limit of a rational function as the variable approaches infinity. While the textbook does not show this explicitly, the most common method used to evaluate this type of limit is to divide all terms by the highest power of *x*, and then applying the limit $\lim_{x \to \infty} \frac{1}{x} = 0$. For example, $\lim_{x \to \infty} \frac{x^2 - 3x + 5}{5x^2 + 4x - 1} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} - \frac{3x}{x^2} + \frac{5}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} - \frac{1}{x^2}} = \lim_{x \to \infty} \frac{1 - \frac{3}{x} + \frac{5}{x^2}}{5 + \frac{4}{x} - \frac{1}{x^2}} = \frac{1 - 0 + 0}{5 + 0 - 0} = \frac{1}{5}.$

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C2 – Solve problems involving asymptotic and end behaviour.** [C, CN, ME, R, T, V]

Students who have achieved this outcome should be able to:

A. Determine the position(s) of the horizontal asymptote(s) of the graph of a rational function, using limit methods.

B. Determine the position(s) of the vertical asymptote(s) of the graph of a rational function, using limit methods.

C. Find and verify end behaviour models for various functions.

D. Sketch the graph of a function with given limits and end behaviour models.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

2.2 (A B C D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C2 – Solve problems involving asymptotic and end behaviour. [C, CN, ME, R, T, V]

Elaboration

Rational functions may have horizontal and vertical asymptotes, depending on the behaviour of the function. Both types of asymptotes may be found by using limit methods.

Horizontal asymptotes of rational functions may be found by evaluating the limit of the function as $x \to \infty$ and $x \to -\infty$. If either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$, then the line $y = b$ will be a horizontal asymptote of the graph of the function $y = f(x)$.

Vertical asymptotes of rational functions may be found by evaluating the limit of the function as *x* approaches a value that makes the denominator of the function equal to zero. If either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$, the line $x = a$ will be a vertical asymptote of the graph of the function $y = f(x)$.

End behaviour models can be used to analyse complicated functions. This allows us to model the behaviour of such a function by a simpler one that acts in virtually the same way. In this way, we can find the limits of certain functions that would be very difficult to find otherwise.

We say that the function *g* is:

- a right behaviour model for *f* if and only if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$;
- a left behaviour model for *f* if and only if $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 1$;
- an end behaviour model for *f* if it provides both a left and right end behaviour model.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C3 – Solve problems involving continuity.** [C, CN, ME, R, T, V]

Students who have achieved this outcome should be able to:

A. Find the points of continuity and discontinuity of a function.

B. Identify examples of discontinuous functions and the types of discontinuities they illustrate, including removable, infinite, jump, and oscillating discontinuities.

C. Determine whether a function is continuous at a point from the definition of continuity.

D. Determine whether a function is continuous on an interval.

E. Determine whether a function is continuous at a point from its graph.

F. Rewrite removable discontinuities by extending or modifying a function.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

2.3 (A B C D E F)

[C] Communication	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving	**[T]** Technology
[CN] Connections		**[R]** Reasoning	**[V]** Visualization

SCO: C3 – Solve problems involving continuity. [C, CN, ME, R, T, V]

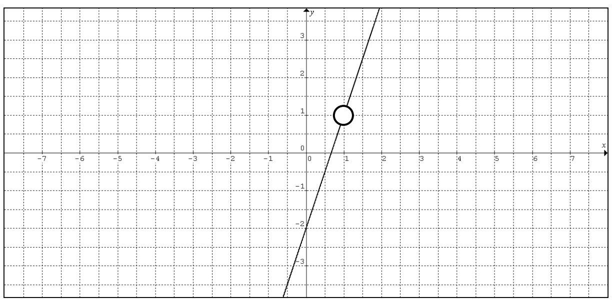
Elaboration

Continuity is an extremely important concept in the study of calculus. A function $y = f(x)$ is continuous at a point *c* in the interior of its domain if the following three conditions are true:

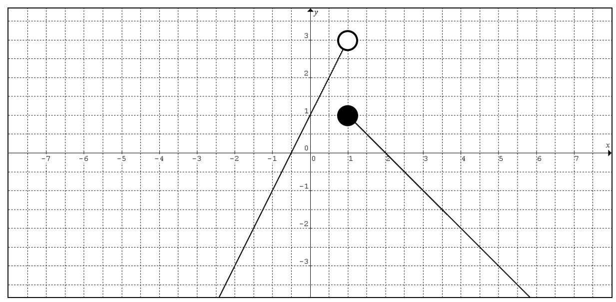
- $f(c)$ exists;
- $\lim_{x \to c} f(x)$ exists;
- $\lim_{x \to c} f(x) = f(c)$.

If a function is not continuous at *c*, then we say that *f* is discontinuous at *c*, and *c* is a point of discontinuity of *f*. There are four types of discontinuities, as illustrated below:

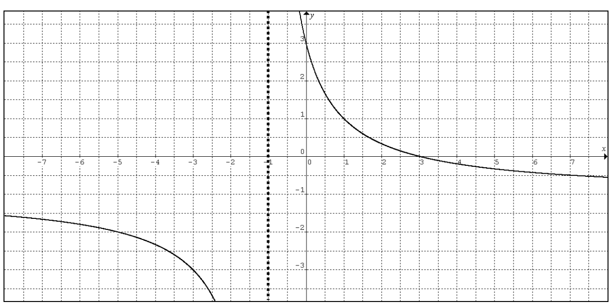
Removable Discontinuity



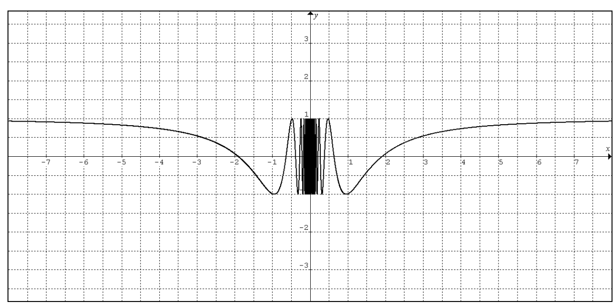
Jump Discontinuity



Infinite Discontinuity



Oscillating Discontinuity



A function with a removable discontinuity can be rewritten by extending or modifying it, which converts it to a continuous function. For example, the function $f(x) = \dfrac{x^2 - 9}{x - 3}$ has a removable discontinuity at $x = 3$. However this function can be extended, since $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6$. As a result, we can say that

$g(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$ is an extended function of $f(x)$, since $g(x)$ is continuous.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C4 – Explore the concept of the derivative as the instantaneous rate of change.** [C, CN, ME, PS, R, T, V]

Students who have achieved this outcome should be able to:

A. Determine the average rate of change of a function over an interval.

B. Demonstrate an understanding that the instantaneous rate of change of a function at a point is the limiting value of a sequence of average rates of change.

C. Determine the slope of a curve at a given point.

D. Determine whether a curve has a tangent line at a given point.

E. Determine the equation of a tangent line to a curve at a given point.

F. Determine the equation of a normal line to a curve at a given point.

G. Calculate and interpret average rates of change drawn from a variety of applications.

H. Solve problems involving instantaneous rates of change drawn from a variety of applications.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

2.4 (A B C D E F G H)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C4 – Explore the concept of the derivative as the instantaneous rate of change. [C, CN, ME, PS, R, T, V]

Elaboration

The average rate of change of a quantity over a period of time is the amount of change divided by the elapsed time. As the amount of time decreases, the average rate of change approaches the instantaneous rate of change at a particular instant.

This concept can be generalized to working with functions. The slope of secant line to the curve $y = f(x)$ joining the points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve, is equal to the expression

$$m_{\text{secant}} = \frac{f(a+h) - f(a)}{h}$$

If the distance between the two points on the curve, *h*, decreases, then this slope approaches the value of the slope of the curve at the point $(a, f(a))$. We can determine this slope by finding the limit of the above expression as $h \to 0$:

$$m_{\text{curve}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The line that passes through this point and has the same slope as the curve is called the tangent line to the curve at that point. The line that passes through this point and is perpendicular to the tangent line is called the normal line to the curve at that point.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C5 – Determine the derivative of a function by applying the definition of derivative.** [CN, ME, R, V]

Students who have achieved this outcome should be able to:

A. Determine the derivative of a function, $f(x)$, by using the limit definition of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ or } f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

B. Given the graph of the derivative of a function, sketch a graph of the function.

C. Given the graph of a function, sketch a graph of its derivative.

D. Determine whether a function is differentiable at a given point.

E. Explain why a function is not differentiable at a given point, and distinguish among corners, cusps, discontinuities, and vertical tangents.

F. Determine all values for which a function is differentiable.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

3.1 (A B C)

3.2 (D E F)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C5 – Determine the derivative of a function by applying the definition of derivative. [CN, ME, R, V]

Elaboration

If we generalize the expression for finding the slope of a curve $y = f(x)$ at the point $(a, f(a))$ to any point on the curve $(x, f(x))$, we get the definition for the derivative, a central concept in calculus.

The derivative of the function *f*, with respect to the variable *x*, is the function f' whose value at *x* is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. An alternate way of defining the derivative is

$$f'(x) = \lim_{h \to a} \frac{f(x) - f(a)}{x - a}$$

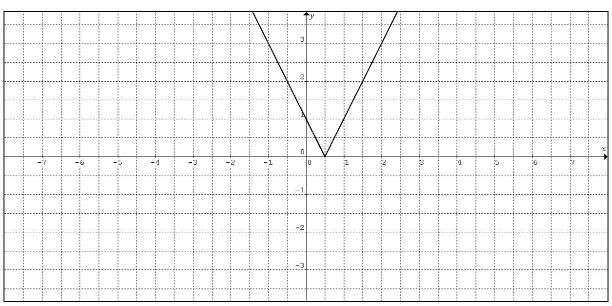
again, providing the limit exists.

Being able to graph the derivative of a function from its graph is a concept that is often stressed in post-secondary calculus courses. To help students understand the process, they should remember the following three basic facts.

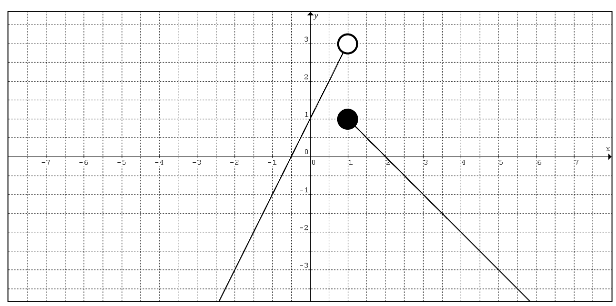
- When the graph of *f* has a positive slope, the graph of f' will be above the *x*-axis.
- When the graph of *f* has zero slope, the graph of f' will be on the *x*-axis.
- When the graph of *f* has a negative slope, the graph of f' will be below the *x*-axis.

It is also important for students to understand when functions are not differentiable. This can occur in four situations, as illustrated below:

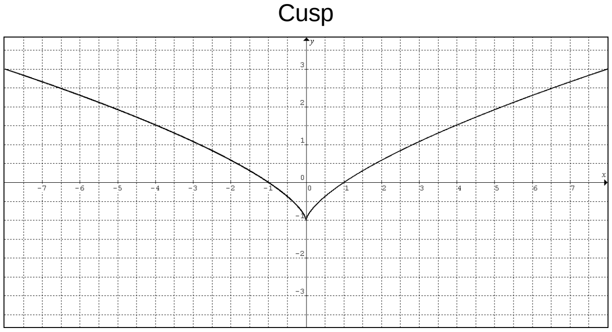
Corner



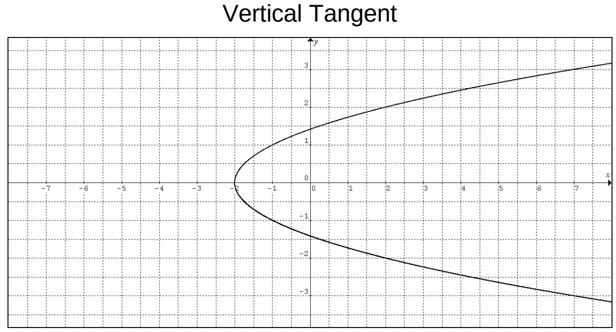
Discontinuity



Cusp



Vertical Tangent



MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C6 – Apply derivative rules to determine the derivative of a function, including:**

- **Constant Rule;**
- **Power Rule;**
- **Constant Multiple Rule;**
- **Sum Rule;**
- **Difference Rule;**
- **Product Rule;**
- **Quotient Rule.**

[C, CN, PS, R]

Students who have achieved this outcome should be able to:

A. Derive the Constant, Power, Constant Multiple, Sum, Difference, Product, and Quotient Rules for determining derivatives.

B. Determine the derivatives of functions, using the Constant, Power, Constant Multiple, Sum, Difference, Product, and Quotient Rules.

C. Determine second and higher-order derivatives of functions.

D. Solve problems involving derivatives drawn from a variety of applications.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

3.3 (A B C D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C6 – Apply derivative rules to determine the derivative of a function, including:

- **Constant Rule;**
- **Power Rule;**
- **Constant Multiple Rule;**
- **Sum Rule;**
- **Difference Rule;**
- **Product Rule;**
- **Quotient Rule.**

[C, CN, PS, R]

Elaboration

There are a number of basic differentiation rules that can be easily derived from the definition of the derivative. They are listed in the table below:

RULES FOR DIFFERENTIATION	
NAME	**LAW**
Constant Rule	$\frac{d}{dx}(c)=0$
Power Rule	$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$
Constant Multiple Rule	$\frac{d}{dx}(cu)=c\frac{du}{dx}$
Sum Rule	$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$
Difference Rule	$\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$
Product Rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

When working with higher-order derivatives, notation can sometimes be a bit confusing for some students. Ensure that they understand that the superscripts that are used for higher-order derivatives, such as $y^{(4)}$ for the fourth derivative of *y*, are not exponents. Also remind students that the notation for higher-order derivatives when using Lagrange's notation includes parentheses, such as $y^{(4)}$ for the fourth derivative, in order to avoid confusion with exponents.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C7 – Use calculus techniques to solve problems involving rates of change, including motion problems involving position, velocity, and acceleration.** [C, CN, PS, R, V]

Students who have achieved this outcome should be able to:

A. Use the derivative to calculate the instantaneous rate of change.

B. Solve problems involving rates of change drawn from a variety of applications.

C. Demonstrate an understanding of the concepts of position, velocity, and acceleration, and the relationships among those concepts.

D. Solve problems involving motion, including position, velocity, and acceleration.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

3.4 (A B C D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C7 – Use calculus techniques to solve problems involving rates of change, including motion problems involving position, velocity, and acceleration. [C, CN, PS, R, V]

Elaboration

As shown earlier in the course, the instantaneous rate of change of a quantity can be analysed by using calculus methods. The instantaneous rate of change of *f* with respect to *x* at *a* is the derivative $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided the limit exists. This concept can be specifically applied to motion along a coordinate line.

Suppose that an object is moving along a coordinate line so that we know its position, *s*, as a function of time, *t*. Since velocity is the rate of change of the position of an object with respect to time, the velocity, $v(t)$, will be the derivative of the position function with respect to time, or $v(t)=\frac{ds}{dt}$. Since acceleration is the rate of change of the velocity of an object with respect to time, the acceleration, or $a(t)$, will be the derivative of the velocity function with respect to time, or $a(t)=\frac{dv}{dt}=\frac{d^2s}{dt^2}$.

Another common example involving rates of change occurs in economics. In a manufacturing operation, the cost of production, $c(x)$, is a function of *x*, the number of units produced. The marginal cost of production is the rate of change of cost with respect to the number of items produced, so $\text{marginal cost}=\frac{dc}{dx}$. Similarly, if the revenue generated by producing *x* units is $r(x)$, the marginal revenue is the rate of change of revenue with respect to the number of items produced, so $\text{marginal revenue}=\frac{dr}{dx}$.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C8 – Find derivatives of trigonometric functions.** [CN, ME, R, V]

Students who have achieved this outcome should be able to:

A. Derive the derivatives of the six basic trigonometric functions.

B. Determine the derivative of a trigonometric function.

C. Solve a problem involving the derivative of a trigonometric function.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

3.5 (A B C)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C8 – Find derivatives of trigonometric functions. [CN, ME, R, V]

Elaboration

The derivatives of the trigonometric functions $y = \sin x$ and $y = \cos x$ can be derived by using the definition of the derivative along with the limits of certain trigonometric expressions that were developed earlier in the course. The derivatives of the four other trigonometric functions can be derived by rewriting each function in terms of $\sin x$ and/or $\cos x$, and then using the Quotient Rule. The derivatives of the six trigonometric functions are:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C9 – Apply the Chain Rule to determine the derivative of a function.** [C, CN, PS, R, V]

Students who have achieved this outcome should be able to:

A. Demonstrate an understanding of the Chain Rule.

B. Determine the derivative of a composite function, using the Chain Rule.

C. Solve a problem involving the derivative of a composite function.

D. Determine the derivative of a relation, using implicit differentiation.

E. Determine the equation of the tangent and normal lines to the graph of a relation at a given point.

F. Determine the second derivative of a relation, using implicit differentiation.

G. Solve problems involving implicit differentiation drawn from a variety of applications.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

4.1 (A B C)

4.2 (D E F G)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C9 – Apply the Chain Rule to determine the derivative of a function. [C, CN, PS, R, V]

Elaboration

The derivative of the function $y = (x+1)^{10}$ can be found by expanding the expression $(x+1)^{10}$, then applying the Sum Rule to the expanded expression. But this is an inefficient method, especially if the outer exponent is very large. Finding the derivative of the function $y = \sin\ (x+1)$ can be found by using the trigonometry identity for the sine of a sum and then applying the appropriate derivative rules to find the answer, but again the method is inefficient.

The Chain Rule can be used to easily find the derivative of composite functions such as these and many others. It states that if two functions *f* and *g* are both differentiable, and $F(x) = (f \circ g)(x) = f[g(x)]$, then

$$F'(x) = (f \circ g)'(x) = f'[g(x)] \bullet g'(x)$$

Because of its structure, the Chain Rule is sometimes referred to as the “Outside-Inside” Rule.

An important case of the Chain Rule applies to a function that is made up of an expression raised to a power. The Power Chain Rule, as it is called, states that if *n* is a real number and $f(u) = u^n$, then

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

It is not always possible for a relation to be written as an explicit function of one variable. In order to find $\frac{dy}{dx}$ in these cases, we must use the process of implicit differentiation, which means to take the derivative of each term in the relation with respect to the independent variable, *x*. Then, solve for $\frac{dy}{dx}$ to get the derivative.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C10 – Solve problems involving inverse trigonometric functions.** [CN, ME, R, V]

Students who have achieved this outcome should be able to:

- **A.** Explain the relationship between each of the trigonometric functions and its corresponding inverse trigonometric function.
- **B.** Explain why inverse trigonometric functions have restricted domains and ranges.
- **C.** Determine the exact value of an expression involving an inverse trigonometric function.
- **D.** Simplify an expression involving an inverse trigonometric function.
- **E.** Determine the domain of an inverse trigonometric function.
- **F.** Sketch the graph of an inverse trigonometric function.
- **G.** Determine the derivative of an inverse trigonometric function.
- **H.** Solve a problem involving the derivative of an inverse trigonometric function.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

1.6 (A B C D E F)

4.3 (G H)

[C] Communication	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving	**[T]** Technology
[CN] Connections		**[R]** Reasoning	**[V]** Visualization

SCO: C10 – Solve problems involving inverse trigonometric functions. [CN, ME, R, V]

Elaboration

In order to determine the inverses of the six trigonometric functions, their domains and ranges must be restricted so that the functions are one-to-one. As a result, the six inverse trigonometric functions are restricted as follows:

FUNCTION	DOMAIN	RANGE
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \csc^{-1} x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

To find the derivatives of the six inverse trigonometric functions, we can rewrite each function by solving for *x* and then using the method of implicit differentiation. As a result, we get the following derivatives:

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C11 – Find limits and derivatives of exponential and logarithmic functions.** [CN, ME, R, V]

Students who have achieved this outcome should be able to:

A. Establish the exponential limit $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, using informal methods.

B. Determine the derivatives of the exponential functions $y = a^x$ and $y = e^x$, and of the logarithmic functions $y = \log_a x$ and $y = \ln x$.

C. Determine the derivative of an exponential function.

D. Determine the derivative of a logarithmic function.

F. Determine the derivative of a function using logarithmic differentiation.

G. Solve a problem involving the derivative of an exponential or a logarithmic function.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

4.4 (A B C D E F G)

[C] Communication	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving	**[T]** Technology
[CN] Connections		**[R]** Reasoning	**[V]** Visualization

SCO: C11 – Find limits and derivatives of exponential and logarithmic functions. [CN, ME, R, V]

Elaboration

The derivatives of the two basic exponential functions and the two basic logarithmic functions are as follows:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}(\ln\ x) = \frac{1}{x}$$

$$\frac{d}{dx}a^x = a^x\ \ln\ a \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x\ \ln\ a}$$

To find the derivative of a function containing a variable in the exponent, we use the process of logarithmic differentiation. This process involves taking the natural logarithm of both sides, taking the derivative of each term with respect to *x*, and then solving for $\frac{dy}{dx}$. This will create an expression which has two different variables in the derivative. To eliminate one variable, substitute the original function for *y* to get the final answer in terms of *x* only.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C12 – Use calculus techniques to sketch the graph of a function.** [C, CN, PS, R, T, V]

Students who have achieved this outcome should be able to:

A. Determine the local and global extreme values of a function.

B. Demonstrate an understanding of the Extreme Value Theorem.

C. Determine the critical and stationary points of a function.

D. Demonstrate an understanding of the Mean Value Theorem.

E. Determine the intervals on which a function is increasing and decreasing.

F. Use the First and Derivative Tests to classify the local extrema of a function.

G. Use the Concavity Test to determine the intervals of concavity of a function.

H. Determine the points of inflection of a function.

I. Determine the key features of the graph of a function, using the techniques of differential calculus, and use these features to sketch the graph without technology.

J. Sketch the graph of a function, using information about its derivative.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

5.1 (A B C)

5.2 (D E)

5.3 (F G H I J)

[C] Communication	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving	**[T]** Technology
[CN] Connections		**[R]** Reasoning	**[V]** Visualization

SCO: C12 – Use calculus techniques to sketch the graph of a function. [C, CN, PS, R, T, V]

Elaboration

When sketching the graph of a function $y = f(x)$, there are a number of steps that should be followed:

A. Determine the domain of the function. It is often helpful to start by determining the set of values of *f* for which $y = f(x)$ is defined.

B. Determine the intercepts of the function. The *y*-intercept is $f(0)$ and this tells us where the curve intersects the *y*-axis. To find the *x*-intercepts, we set $y = 0$ and solve for *x*. If the equation is difficult to solve for *x*, technology can be used to approximate the *x*-intercepts.

C. Determine whether the graph is symmetric.

- If $f(-x) = f(x)$ for all *x* in the domain, then *f* is an even function, which means that the curve is symmetric about the *y*-axis.
- If $f(-x) = -f(x)$ for all *x* in the domain, then *f* is an odd function, which means that the curve is symmetric about the origin.
- If $f(x+p) = f(x)$ for all *x* in the domain, where *p* is a positive constant, then *f* is periodic, which means that the curve repeats over the interval *p*.

D. Determine whether the graph has any asymptotes.

- If there exists a line $y = b$ such that either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, or both, then $y = b$ is called a horizontal asymptote of the curve $y = f(x)$.
- If there exists a line $x = a$ such that either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then $x = a$ is called a vertical asymptote of the curve $y = f(x)$.

E. Determine the intervals of increase or decrease. Find the critical values of $y = f(x)$ using its derivative $y = f'(x)$, that is, the values where $f'(x) = 0$ or $f'(x)$ does not exist. The function will be increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$.

F. Determine the local maximum and local minimum values. Use the First Derivative Test to determine whether the critical values are local maximum or local minimum values. If f' changes from negative to positive at *c* and $f(c)$ exists, then $f(c)$ is a local minimum. If f' changes from positive to negative at *c* and $f(c)$ exists, then $f(c)$ is a local maximum.

G. Determine the intervals of concavity and the points of inflection. Find $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. An inflection point will occur where $f''(x) = 0$, providing $f(x)$ exists, and the concavity is different on either side of the point.

H. Sketch the graph. A table of values should be used that includes all of the intercepts, critical values, and points of inflection, and the table should extend at least two units beyond all of these points to the left and to the right. Use the information from the table of values and from Steps A-G to sketch the graph.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C13 – Use calculus techniques to solve optimization problems.** [C, CN, ME, PS, R]

Students who have achieved this outcome should be able to:

A. Determine the equation of the objective function to be optimized in an optimization problem.

B. Determine the equations of any parameters necessary in an optimization problem.

C. Solve an optimization problem drawn from a variety of applications, using calculus techniques.

D. Interpret the solution(s) to an optimization problem.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

5.4 (A B C D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C13 – Use calculus techniques to solve optimization problems. [C, CN, ME, PS, R]

Elaboration

A problem which seeks to find the maximum or minimum answer to a situation involving continuous variables is called an optimization problem. The steps in solving an optimization problem are as follows:

A. Understand the problem. The first step is to read the problem carefully until it is clearly understood. Ask yourself: *What is the unknown? What are the given quantities? What are the given conditions?*

B. Draw a diagram, if necessary. In many optimization problems, it is useful to draw a diagram and identify the given and required quantities on the diagram.

C. Introduce notation. Assign a symbol to the quantity that is to be maximized or minimized. This quantity is called the objective function. Also, select symbols for other unknown quantities and label the diagram with these symbols. It may help to use variables that represent the quantities in the problem, such as P for profit, or r for radius.

D. Express the objective function in terms of the symbols introduced above. This is the function that ultimately will be maximized or minimized.

E. Express the objective function in terms of only one variable. If the objective function contains more than one variable on the right hand side, it will be necessary to rewrite the objective function so that the right-hand side contains only one variable. In order to do this, look for relationships among the variables that were introduced above. Each of these relationships is called a parameter. Use these parameters to eliminate all but one of the variables on the right hand side of the objective function. Also note the domain of this function in order to eliminate any possible extraneous solutions.

F. Use calculus techniques to find the absolute maximum or minimum value of the objective function. Find the critical values of the objective function by taking its derivative and determining its critical values. Use the First Derivative Test to determine whether each critical value is a local maximum or minimum value. Then, use this information to find either the absolute maximum or absolute minimum value of the objective function.

G. Interpret the solution. Ensure that the solution is in the domain of the objective function. If it is, then translate your mathematical result into the problem setting (with appropriate units).

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C14 – Use linearization and Newton's Method to solve problems.** [C, CN, ME, PS, R]

Students who have achieved this outcome should be able to:

A. Use linearization to approximate a numerical expression.

B. Determine the differential of a function.

C. Solve an linearization problem drawn from a variety of applications.

D. Use Newton's Method to approximate the solution(s) of an equation.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

5.5 (A B C D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C14 – Use linearization and Newton's Method to solve problems. [C, CN, ME, PS, R]

Elaboration

If *f* is differentiable at $x = a$, then the equation of the tangent line

$$L(x) = f(a) + f'(a)(x - a)$$

is defined as the linearization of *f* at *a*. Linearization can be used to find an approximation of a value that is difficult to calculate by hand but is close to a known value, such as $\sqrt{145}$ or $\cos 3.1$.

Leibniz used the notation $\frac{dy}{dx}$ to represent the derivative of *y* with respect to *x*. The notation looks like a quotient of real numbers, but it is really a limit of quotients in which both numerator and denominator are infinitesimally close to zero. Since we say that $\frac{dy}{dx} = f'(x)$, and $\frac{dy}{dx}$ acts as a quotient, we can define the differential *dy* as

$$dy = f'(x)dx$$

This implies that *dy* is a dependent variable that depends on both *x* and *dx*. The differential, *dy*, represents the difference between the linearization and the true function value at a given point.

Newton's Method is a numerical technique for approximating a zero of a function with the zeros of its linearizations. Under favourable circumstances, the zeros of the linearizations converge rapidly to an accurate approximation. Many calculators use this method to find zeros because it applies to a wide range of functions and gets accurate results in only a few steps. The procedure is as follows:

A. Guess a first approximation to a solution of the equation $f(x) = 0$. Try to be as close to the actual solution as possible.

B. Use the first approximation to get a second, the second approximation to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Please note that Newton's method may fail in certain circumstances. If $f'(x_1) = 0$, it will fail because the formula will involve division by zero. In that case, choose another starting value. The other common instance in which Newton's Method will not work is when the starting value is too far from the actual solution. In that case, choose a closer starting value. There are also other less common instances in which Newton's Method will not work, depending on the complexity of the function.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C15 – Solve problems involving related rates.** [C, CN, ME, PS, R]

Students who have achieved this outcome should be able to:

A. Develop a mathematical model for a related rates problem.

B. Solve a problem involving related rates, drawn from a variety of applications.

C. Interpret the solution to a related rates problem.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

5.6 (A B C)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C15 – Solve problems involving related rates. [C, CN, ME, PS, R]

Elaboration

A related rates problem is a problem which involves finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The steps in solving a related rates problem are as follows:

A. Understand the problem. In particular, identify the variable whose rate of change you seek and the variable(s) whose rates of change you know.

B. Develop a mathematical model of the problem. Draw a diagram and label the parts that are important to the problem with symbols that have been assigned to each quantity. Be sure to distinguish constant quantities from variables that change over time. Only constant quantities can be assigned numerical values at the start.

C. Write an equation relating the variable whose rate of change you seek with the variable(s) whose rates of change you know. The formula is often geometric, but it should come from a scientific explanation.

D. Differentiate both sides of the equation implicitly with respect to time, *t*. Be sure to follow all the differentiation rules. The Chain Rule will be especially critical, as you will be differentiating with respect to time, t.

E. Substitute values for any quantities that depend on time. Notice that it is only safe to do this after the differentiation step. Substituting too soon makes changeable variables behave like constants, with zero derivatives.

G. Interpret the solution. Translate your mathematical result into the problem setting (with appropriate units), and decide whether the result makes sense.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C16 – Determine the indefinite integral of a function.** [C, CN, PS, R]

Students who have achieved this outcome should be able to:

A. Determine the general antiderivative of a function.

B. Determine the indefinite integral of the following functions:

- $y = x^n$;
- $y = \sin x$;
- $y = \cos x$;
- $y = e^x$;
- $y = \frac{1}{x}$.

C. Demonstrate an understanding of the properties for indefinite integrals.

D. Use a substitution to determine the indefinite integral of a function.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

6.3.2 (A B C)

7.2 (D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C16 – Determine the indefinite integral of a function. [C, CN, PS, R]

Elaboration

This specific curriculum outcome will highlight the connection between antidifferentiation and differentiation by applying the Fundamental Theorem of Calculus.

When antidifferentiating, the Power Rule for Antidifferention is often used. If $f(x) = x^n$, then the general antiderivative, $F(x)$, is

$$F(x) = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

where C represents an arbitrary constant.

In addition, the following is a list of the antiderivative of some other important functions:

FUNCTION	ANTIDERIVATIVE
$y = \cos\ x$	$y = \sin\ x + C$
$y = \sin\ x$	$y = -\cos\ x + C$
$y = e^x$	$y = e^x + C$
$y = \frac{1}{x}$	$y = \ln\ \lvert x \rvert + C,\ x > 0$

Because of the Fundamental Theorem of Calculus, the concepts of antidifferentiation and indefinite integration are essentially the same thing. As a result, the solution to an indefinite integral problem can always be easily verified by taking the derivative of the solution. If the derivative of the solution is equal to the function that is being integrated, then the solution is correct.

A common method for finding an indefinite integral is by using a substitution. If the function that is being integrated can be written as the product of an expression and a multiple of its differential, then the substitution method can be used.

For example, consider $\int 3x\sqrt{9-3x^2}\,dx$. If we let $u = 9 - 3x^2$, then $du = -6x\,dx$. Since we can write the original integral as $\int\left(-\frac{1}{2}u^{1/2}\right) \bullet du$, we can use the Substitution Method this case: $\int 3x\sqrt{9-3x^2}\,dx = \int\left(-\frac{1}{2}u^{1/2}\right) \bullet du =$ $-\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C = -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}\left(9-3x^2\right)^{3/2} + C.$

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C17 – Determine the definite integral of a function.** [C, CN, PS, R]

Students who have achieved this outcome should be able to:

A. Estimate an area using a finite sum.

B. Convert a Riemann sum to a definite integral.

C. Evaluate a definite integral using an area formula.

D. Solve a problem using the rules for definite integrals.

E. Understand the relationship between the derivative and definite integral as expressed in both parts of the Fundamental Theorem of Calculus.

F. Calculate the definite integral of a function over a closed interval $[a,b]$.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

6.1 (A)

6.2 (B C)

6.3.1 (D)

6.4 (E F)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C17 – Determine the definite integral of a function. [C, CN, PS, R]

Elaboration

The area under a curve on an interval $[a,b]$ can be estimated by using the Rectangular Approximation Method (RAM). The height of each rectangle can be determined by using a left-hand point (LRAM), a midpoint (MRAM), or a right-hand point (RRAM) in each subinterval. Then, in each case, the sum of the rectangles is used as an approximation to the area of the curve. As the width of the rectangles gets smaller, the approximations get closer to the actual area of the region in question.

We can then define the definite integral of a continuous function on $[a,b]$ as follows. If *f* is continuous on $[a,b]$, and $[a,b]$ is partitioned into *n* subintervals of equal length $\Delta x = \frac{b-a}{n}$, then the definite integral of *f* over $[a,b]$ is given by

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

where each c_k is chosen arbitrarily in the *k*th interval. We write the definite integral of *f* on $[a,b]$ as $\int_a^b f(x)dx$. It represents the area under $f(x)$ and above the *x*-axis on $[a,b]$.

The Fundamental Theorem of Calculus can be used to show that derivatives and integrals are essentially inverse processes. In fact, the Fundamental Theorem of Calculus is used to evaluate definite integrals, by using the formula

$$\int_a^b f(x)dx = F(b) - F(a)$$

where *F* is any antiderivative of *f* on $[a,b]$.

MAT611B – Topic: Calculus (C)

GCO: Develop introductory calculus reasoning.

SCO: **C18 – Solve problems that involve the application of the integral of a function.** [C, CN, PS, R, V]

Students who have achieved this outcome should be able to:

A. Determine the area under a function, and above the *x*-axis, from $x = a$ to $x = b$.

B. Determine the area between two functions.

Section(s) in Calculus: Graphical, Numerical, Algebraic *that address the specific curriculum outcome with relevant Achievement indicators in brackets:*

8.2 (A B)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: C18 – Solve problems that involve the application of the integral of a function. [C, CN, PS, R, V]

Elaboration

The most common application of the definite integral is its use to find the area between two curves. If *f* and *g* are continuous throughout $[a,b]$, then the area between the curves $y=f(x)$ and $y=g(x)$ from *a* to *b* is

$$A=\int_a^b\left[f(x)-g(x)\right]dx$$

where $f(x)>g(x)$ on (a,b).

As an example, to find the area between $y=x^2+1$ and $y=1-x^2$ from $x=0$ to $x=1$, it would be found by evaluating the following definite integral:

$$\begin{aligned}A&=\int_0^1\left[\left(x^2+1\right)-\left(1-x^2\right)\right]dx\\&=\int_0^1 2x^2\,dx\\&=\left[\frac{2}{3}x^3\right]_0^1\\&=\left[\frac{2}{3}(1)^3\right]-\left[\frac{2}{3}(0)^3\right]\\&=\frac{2}{3}-0\\&=\frac{2}{3}\end{aligned}$$

Curriculum Guide Supplement

This supplement to the *Prince Edward Island MAT611B Mathematics Curriculum Guide* is designed to parallel the primary resource.

For each of the chapters in the text, an approximate timeframe is suggested to aid teachers with planning. The timeframe is based on a total of 80 classes, each with an average length of 75 minutes:

CHAPTER	SUGGESTED TIME
Unit 1 – Limits and Continuity	12 classes
Unit 2 – Derivatives I	15 classes
Unit 3 – Derivatives II	15 classes
Unit 4 – Applications of Derivatives	20 classes
Unit 5 – Integrals and Their Applications	18 classes

Each chapter of the text is divided into a number of sections. In this document, each section is supported by a one-page presentation, which includes the following information:

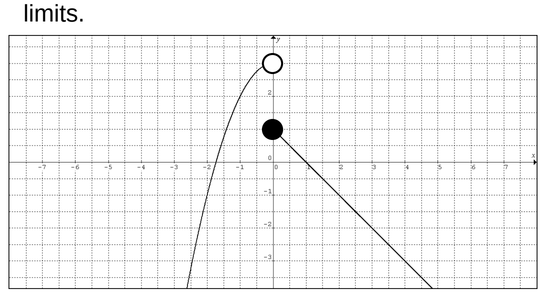
- the name and pages of the section in *Calculus: Graphical, Numerical, Algebraic*;
- the specific curriculum outcome(s) and achievement indicator(s) addressed in the section (see the first half of the curriculum guide for an explanation of symbols);
- the student expectations for the section, which are associated with the SCO(s);
- the new concepts introduced in the section;
- other key ideas developed in the section;
- suggested problems in the primary resource, *Calculus: Graphical, Numerical, Algebraic*;
- suggested problems in the secondary resource, *Calculus* [7E];
- possible instructional and assessment strategies for the section.

UNIT 1
LIMITS AND CONTINUITY

SUGGESTED TIME

12 classes

Section 2.1 – Rates of Change and Limits (pp. 59-69)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C1 (A B C D E F G H I J)** **After this lesson, students will be expected to:** • calculate average and instantaneous speeds • calculate limits for function values and apply the properties of limits • use the Sandwich Theorem to find certain limits indirectly **After this lesson, students should understand the following concepts:** • **average rate of change –** for a function $y = f(x)$, the average rate of change of *y* with respect to *x* over the interval $x \in [a,b]$ is $$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$ • **average speed –** the distance travelled by an object divided by the elapsed time • **instantaneous speed –** the speed of an object at a given instant in time • **limit –** a function *f* has a limit *L* as *x* approaches *a*, written $\lim_{x \to a} f(x) = L$, provided that the values of $f(x)$ get closer and closer to *L*, as *x* gets closer and closer to *a*, from both sides of *a* • **right-hand limit –** a limit, written $\lim_{x \to a^+} f(x)$, which is read "the limit of $f(x)$ as *x* approaches *a* from the right"; used to determine the behaviour of a function, $f(x)$, to the right of $x = a$ • **left-hand limit –** a limit, written $\lim_{x \to a^-} f(x)$, which is read "the limit of $f(x)$ as *x* approaches *a* from the left"; used to determine the behaviour of a function, $f(x)$, to the left of $x = a$ • **two-sided limit –** a limit at an interior point of a function's domain **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 66-69:** #1-70 **Suggested Problems in *Calculus* [7E]:** • **pp. 59-61:** #1-26 • **pp. 69-71:** #1-34	**Possible Instructional Strategies:** • The function $y = \frac{\sin x}{x}$ can provide an excellent introduction to the idea of limits. Using a graphing calculator, students can see that this function approaches 1 as *x* approaches 0. • Many students will have trouble with limits involving the indeterminate form $\frac{0}{0}$. Remind them that when this happens, the function has to be rewritten so that the actual value of the limit can be found. **Possible Assessment Strategies:** • An object dropped from rest from the top of a cliff falls $y = 4.9t^2$ metres in the first *t* seconds. a. Find the average speed during the first 3 seconds of fall. b. Find the speed of the object 3 seconds after it has been dropped. • Determine each of the following limits. a. $\lim_{x \to 1} (2x^2 + 3x + 4)$ b. $\lim_{x \to 5} \frac{x^2 - 6x + 5}{x - 5}$ c. $\lim_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$ d. $\lim_{x \to 0} \frac{(-5 + h)^2 - 25}{h}$ e. $\lim_{x \to 0} \frac{\sin 4x}{\sin 6x}$ • Use the graph to determine each of the following limits.  a. $\lim_{x \to 0^-} f(x)$ b. $\lim_{x \to 0^+} f(x)$ c. $\lim_{x \to 0} f(x)$

Section 2.2 – Limits Involving Infinity (pp. 70-77)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C1 (K L)** • **C2 (A B C D)** **After this lesson, students will be expected to:** • find and verify end behaviour models for various functions • calculate limits as $x \to \pm\infty$, and identify vertical and horizontal asymptotes **After this lesson, students should understand the following concepts:** • **horizontal asymptote –** the line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, or both • **vertical asymptote –** the line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$ • **right end behaviour model –** the function *g* is a right end behaviour model for *f* if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ • **left end behaviour model –** the function *g* is a left end behaviour model for *f* if and only if $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$ • **end behaviour model –** the function *g* is an end behaviour model for *f* if it is both a left end and a right end behaviour model for *f* **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 76-77:** #1-48 **Suggested Problems in *Calculus* [7E]:** • **pp. 59-61:** #29-37 • **pp. 234-237:** #1-15	**Possible Instructional Strategies:** • A discussion of the function $y = \frac{1}{x}$ provides a meaningful introduction to both limits as $x \to \pm\infty$ and infinite limits as $x \to 0$. • The mathematical meaning of the symbol ∞ should be understood in the context of the phrase $x \to \infty$, meaning that for function in *x*, the *x*-value increases without bound. Students should understand that ∞ does not represent a real number. • The end behaviour of three types of rational functions, where the degree of the numerator is less than, equal to, and greater than the degree of the denominator, should be discussed. **Possible Assessment Strategies:** • Determine the horizontal and vertical asymptotes of the graph of $y = \frac{2x^2}{x^2 - 1}$. • Determine each of the following limits. a. $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ b. $\lim_{x\to 3^+} \frac{x}{3 - x}$ c. $\lim_{x\to-\infty} \frac{5x^3 - 2x^2 + 5x - 1}{3 - 4x + 9x^2 + 2x^3}$ • For each of the following, find a power function end behaviour model for *f*. a. $f(x) = 2x^2 - 5x + 1$ b. $f(x) = \frac{x^3 - 5}{2x^2 - 3x + 2}$ c. $f(x) = \frac{9 - x}{x + 1}$ • Find a simple basic function as an end behaviour model for $y = 3x^2 - \cos x$.

Section 2.3 – Continuity (pp. 78-86)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C3 (A B C D E F)** **After this lesson, students will be expected to:** • identify the intervals upon which a given function is continuous and understand the meaning of a continuous function • remove removable discontinuities by extending or modifying a function • apply the Intermediate Value Theorem, and the properties of algebraic combinations and composites of continuous functions **After this lesson, students should understand the following concepts:** • **continuous at an interior point –** a function $f(x)$ is continuous at an interior point $x = c$ if $\lim_{x \to c} f(x) = f(c)$ • **continuous at a left endpoint –** a function $f(x)$ is continuous at a left endpoint $x = a$ of its domain if $\lim_{x \to a^+} f(x) = f(a)$ • **continuous at a right endpoint –** a function $f(x)$ is continuous at a right endpoint $x = b$ of its domain if $\lim_{x \to b^-} f(x) = f(b)$ • **discontinuous –** a function is discontinuous at $x = a$ if it is not continuous at $x = a$ • **point of discontinuity –** a function has a point of discontinuity at $x = a$ if it is not continuous at $x = a$ • **removable discontinuity –** a function *f* has a removable discontinuity at $x = a$ if $\lim_{x \to a} f(x) = L$, and either $f(a) \neq L$ or $f(a)$ does not exist • **jump discontinuity –** a function *f* has a jump discontinuity at $x = a$ if $\lim_{x \to a^-} f(x) = L$, $\lim_{x \to a^+} f(x) = M$, and $L \neq M$ • **infinite discontinuity –** an infinite discontinuity occurs where a graph has a vertical asymptote • **oscillating discontinuity –** a point near where the values of a function oscillate too much for the function to have a limit	**After this lesson, students should understand the following concepts:** • **continuous extension –** a function identical to another function except that is it continuous at one or more points where the other function is not • **continuous on an interval –** a function that is continuous at each point in the interval • **continuous function –** a function that is continuous at every number in its domain • **Intermediate Value Property –** a function that has the property of never taking on two values without taking on all values in between those two values • **connected graph –** a graph that can be drawn as a single, unbroken curve **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 84-86:** #1-44 **Suggested Problems in *Calculus* [7E]:** • **pp. 90-93:** #1-43 **Possible Instructional Strategies:** • Explain to the class that an intuitive way to show the continuity of a function is by being able to draw it without lifting the pencil from the paper. • It is important to visually show each type of discontinuity. Names for the different types of discontinuities should be used consistently throughout the course. **Possible Assessment Strategies:** • Find, and identify, the points of discontinuity of each of the following functions. a. $y = \frac{1}{x+2}$ b. $y = \begin{cases} 1 - x^2, & x < 1 \\ x, & x \geq 1 \end{cases}$ • For each of the following, give a formula for the extended function that is continuous at the indicated *x*-value. a. $y = \frac{x^2 - x - 2}{x - 2}$, $x = 2$ b. $y = \frac{\cos x - 1}{x}$, $x = 0$ • Sketch a possible graph for a function *f* for which $f(2) = 3$, but $\lim_{x \to 2} f(x) = 1$.

Section 2.4 – Rates of Change and Tangent Lines (pp. 87-95)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C4 (A B C D E F G H)** **After this lesson, students will be expected to:** • apply directly the definition of the slope of a curve in order to calculate slopes • find the equations of the tangent line and the normal line to a curve at a given point • find the average rate of change of a function **After this lesson, students should understand the following concepts:** • **slope of a curve –** the slope of the curve $y=f(x)$ at the point $(a,f(a))$ is $f'(a)$, provided *f* is differentiable at *a* • **tangent line to a curve –** if a function $y=f(x)$ is differentiable at $x=a$, then a line is tangent to the graph of *f* at $(a,f(a))$ provided that the line passes through $(a,f(a))$ and the slope of the line is $f'(a)$ • **difference quotient –** either of the expressions $\frac{f(a+h)-f(a)}{h}$ or $\frac{f(x)-f(a)}{x-a}$ • **normal line to a curve –** the normal line to a curve at a point is the line perpendicular to, and intersecting, the tangent line to the curve at that point • **instantaneous rate of change –** for a function $y=f(x)$, the instantaneous rate of change of *y* with respect to *x* over the interval $x\in[a,b]$ is $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 92-95:** #1-12, 19-36 **Suggested Problems in *Calculus* [7E]:** • **pp. 110-113:** #1-20	**Possible Instructional Strategies:** • The average rates of change introduced in this section provide an application related to limits of rational functions. They are also related to derivatives, although the word *derivative* is not used in this section. • Many students are prone to algebraic errors when calculating the slopes of curves. Encourage students to understand their own mistakes so that they will be less likely to make those mistakes in the future. **Possible Assessment Strategies:** • Find the average rate of change of each function over the indicated interval. a. $y=4x-3x^2$, $[2,3]$ b. $y=\sqrt{x}$, $[0,2]$ c. $y=\frac{2x+1}{x+2}$, $[1,3]$ • For each function at the indicated *x*-value, find ➢ the slope of the curve; ➢ the equation of the tangent line; ➢ the equation of the normal line. a. $y=4x-x^2$; $x=1$ b. $y=x-x^3$; $x=0$ c. $y=\frac{2}{x+1}$; $x=-2$ • Find the instantaneous rate of change of the position function $y=t^2-8t$, in metres, at $t=2$ seconds. • What is the rate of change of the area of a square with respect to the length of a side when the length is 5 cm?

UNIT 2
DERIVATIVES I

SUGGESTED TIME
15 classes

Section 3.1 – Derivative of a Function (pp. 99-108)

ELABORATIONS & SUGGESTED PROBLEMS

Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed:

- **C5 (A B C)**

After this lesson, students will be expected to:

- calculate slopes and derivatives, using the definition of the derivative
- graph *f* from the graph of f', and graph f' from the graph of *f*

After this lesson, students should understand the following concepts:

- **derivative of a function at a point *a* –** the value of the derivative of a function $y = f(x)$ at $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  provided the limit exists
- **derivative of a function with respect to *x* –** the derivative of a function $y = f(x)$ is a function defined as

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  provided the limit exists
- **differentiable function –** a function *f* is said to be differentiable at *a* if $f'(a)$ exists
- **differentiable on an interval –** a function *f* is said to be differentiable on an interval if it is differentiable at every number in the interval
- **right-hand derivative –** the derivative defined by a right-hand limit
- **left-hand derivative –** the derivative defined by a left-hand limit

Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:

- **pp. 105-108:** #1-23, 27-28

Suggested Problems in *Calculus* [7E]:

- **pp. 122-126:** #1-32

POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES

Possible Instructional Strategies:

- A good way to introduce this section is by presenting the graph of a function and showing how the slopes of secant lines approaches a limit corresponding to the slope of a tangent line at a given point.
- Students should learn to calculate derivatives using the definition. This can be done in two stages. First, calculate the derivative at a particular value $x = a$. Then, generalize the process to find the derivative at a general point *x*. The different notations for the derivative should be discussed.
- When finding the derivative using the definition, students often make errors when simplifying the numerator of the difference quotient. When $f(x)$ is a polynomial or a rational function, *h* is always a factor in the numerator of the simplified expression.

Possible Assessment Strategies:

- Find the value of the derivative of each of the following functions using the definition of derivative at the indicated value of *a*.

  a. $f(x) = 5x - 9x^2;\ a = 1$

  b. $f(x) = \dfrac{1}{\sqrt{x}};\ a = 4$

  c. $f(x) = \dfrac{x^2 - 1}{2x - 3};\ a = 2$

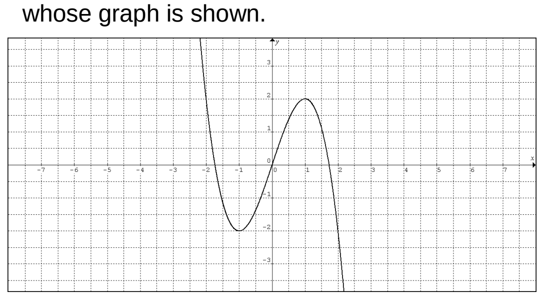
- Differentiate each of the following functions using the definition of derivative.

  a. $f(x) = x^2 - 2x^3$

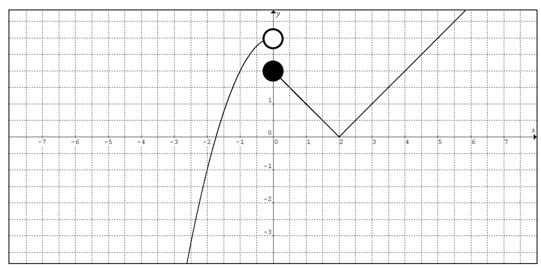
  b. $f(x) = \sqrt{9 - x}$

  c. $f(x) = \dfrac{1 - 2x}{3 + x}$

- Sketch the graph of the derivative of the function whose graph is shown.



Section 3.2 – Differentiability (pp. 109-115)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C5 (D E F)** **After this lesson, students will be expected to:** • determine where a function is not differentiable and distinguish among corners, cusps, discontinuities, and vertical tangents **After this lesson, students should understand the following concept:** • **local linearity –** a feature of all smooth curves; any smooth curve will appear to be linear if you focus on a small enough (local) domain **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 114-115:** #1-16, 31-35 **Suggested Problems in *Calculus* [7E]:** • **pp. 122-126:** #35-38	**Possible Instructional Strategies:** • Introduce this lesson with an informal discussion of what it means for a function to be differentiable or non-differentiable at a point. Give several examples to illustrate corners, cusps, vertical tangents, and discontinuities. • Students must know when $f'(a)$ fails to exist. For example, let $f(x)=\begin{cases}2x+1, & x\le 2\\ \frac{1}{2}x^2+4, & x>2\end{cases}$. Note that *f* is not continuous at $x=2$, so $f'(2)$ does not exist. **Possible Assessment Strategies:** • At what point(s) is the graph of the following function not differentiable? Why?  • Each of the following functions fails to be differentiable at $x=1$. Determine whether the problem is a corner, cusp, vertical tangent, or discontinuity. a. $y=(x-1)^{2/3}$ b. $y=\lvert x-1\rvert$ c. $y=\dfrac{1}{x-1}$ d. $y=(x-1)^{1/3}$ • Determine all values of *x* for which each function is not differentiable. a. $f(x)=\dfrac{x}{x-5}$ b. $f(x)=\sqrt[3]{2x-6}$ c. $f(x)=\sin\ \lvert x+2\rvert$ d. $f(x)=\begin{cases}-1, & x<0\\ 1, & x\ge 0\end{cases}$

Section 3.3 – Rules for Differentiation (pp. 116-126)

ELABORATIONS & SUGGESTED PROBLEMS

Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed:

- **C6 (A B C D)**

After this lesson, students will be expected to:

- use the rules of differentiation to find derivatives, including second and higher order derivatives
- use the derivative to calculate the instantaneous rate of change

After this lesson, students should understand the following concept:

- ***n*th derivative –** an expression indicating that a function is differentiated *n* times; denoted by $\frac{d^n y}{dy^n}$ or $y^{(n)}$

RULES FOR DIFFERENTIATION	
NAME	**LAW**
Constant Rule	$\frac{d}{dx}(c)=0$
Power Rule	$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$
Constant Multiple Rule	$\frac{d}{dx}(cu)=c\frac{du}{dx}$
Sum Rule	$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$
Difference Rule	$\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$
Product Rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:

- **pp. 124-126:** #1-47

Suggested Problems in *Calculus* [7E]:

- **pp. 136-139:** #1-62

POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES

Possible Instructional Strategies:

- Learning the differentiation rules and their correct use is essential for all that follows in this course, including the applications of derivatives and the antidifferentiation process. The Quotient Rule can be especially difficult for some students to remember. The following mnemonic device can be used to remember the Quotient Rule:

$$d\left(\frac{\text{hi}}{\text{low}}\right)=\frac{\text{low di hi}-\text{hi di low}}{\text{low}^2}$$

- Many students make errors when using the rules for differentiation, both in applying the rules and simplifying answers. Encourage them to always check their work.
- When applying the Quotient Rule, students often interchange the terms in the numerator of the derivative when subtracting. Students who cannot remember the order of the terms should apply the Quotient Rule to a simple function such as $f(x)=\frac{x}{1}$ in order to determine the correct order.

Possible Assessment Strategies:

- Find the derivative of each of the following functions.

  a. $f(x)=\frac{1}{2}x^6-3x^4+x$

  b. $f(x)=\left(1+x+x^2\right)\left(2-x^4\right)$

  c. $f(x)=\left(x^3-2x\right)\left(x^{-4}+x^{-2}\right)$

  d. $f(x)=\frac{x+1}{x^2+x-2}$

- For each of the following, find an equation for the line tangent to the curve at the indicated *x*-value.

  a. $y=(1+2x)^2,\ \ x=1$

  b. $y=\frac{3x+1}{x^2+1},\ \ x=-1$

- Find the first three derivatives of each of the following functions.

  a. $y=x^4-3x^3+16x$

  b. $y=\frac{2x-1}{x}$

- Find equations of both lines that are tangent to the curve $y=1+x^3$ and have a slope 12.

Section 3.4 – Velocity and Other Rates of Change (pp. 127-140)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C7 (A B C D)** **After this lesson, students will be expected to:** • use derivatives to analyse straight line motion and solve other problems involving rates of change **After this lesson, students should understand the following concepts:** • **displacement –** the difference between the path of the initial and final position covered by a moving object; displacement is calculated using the formula $\Delta s = f(t+\Delta t) - f(t)$ • **average velocity –** displacement divided by time travelled; average velocity is calculated using the formula $\bar{v} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t}$ • **instantaneous velocity –** the derivative of a position function with respect to time; instantaneous velocity is calculated using the formula $v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t}$ • **speed –** the magnitude of velocity, $v(t)$, $\text{speed} = \lvert v(t) \rvert = \left\lvert \frac{ds}{dt} \right\rvert$ • **acceleration –** the rate of change of the velocity of an object with respect to time $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ • **sensitivity of a function –** a measure of the rate of change of a function • **marginal cost –** the instantaneous rate of change of cost with respect to the number of items produced; the marginal cost is the derivative of the cost function • **marginal revenue –** the instantaneous rate of change of revenue with respect to the number of items produced; the marginal revenue is the derivative of the revenue function **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 135-140:** #1-8, 13-16, 19-30 **Suggested Problems in *Calculus* [7E]:** • **pp. 136-139:** #63-65 • **pp. 173-176:** #1-10	**Possible Instructional Strategies:** • The acceleration due to gravity can be either a positive or a negative number, depending on how one defines a coordinate system. Discuss the importance of choosing a convention and remaining consistent within any particular application. • The discussion of marginal cost and marginal revenue is an excellent way to show that many rates of change do not involve time or position. **Possible Assessment Strategies:** • The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$, where t is measured in seconds and s in metres. a. Find the velocity at time t. b. What is the velocity after 2 s? after 4 s? c. When is the particle at rest? d. When is the particle moving in the positive direction? in the negative direction? e. Find the acceleration at time t and after 4 s. f. When is the particle speeding up? When is the particle slowing down? • The height, in metres, of a projectile shot vertically upward from ground level with an initial velocity of 24.5 m/s is $h = 24.5t - 4.9t^2$, after t seconds. a. Find the velocity after 2 s and after 4 s. b. When does the projectile reach its maximum height? c. What is the maximum height? d. When does it hit the ground? e. With what velocity does it hit the ground? • If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume of water remaining in the tank after t minutes as $V = 5000\left(1 - \frac{1}{40}t\right)^2$, where $0 \le t \le 40$. a. Find the rate at which the water is draining from the tank after 5 min, 10 min, 20 min, and 40 min. b. At what time is the water flowing out the fastest? the slowest?

Section 3.5 – Derivatives of Trigonometric Functions (pp. 141-147)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C8 (A B C)** **After this lesson, students will be expected to:** • use the rules for differentiating the six basic trigonometric functions **After this lesson, students should understand the following concept:** • **jerk –** the derivative of an acceleration function with respect to time $$j(t)=\frac{da}{dt}=\frac{d^2v}{dt^2}=\frac{d^3s}{dt^3}$$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 146-147:** #1-36 **Suggested Problems in *Calculus* [7E]:** • **pp. 146-148:** #1-24	**Possible Instructional Strategies:** • Have students investigate the behaviour of the graph of the derivative of $y=\sin x$. By doing this, students should recognize the graph of $y=\cos x$. • The rules for differentiating $y=\sin x$ and $y=\cos x$ can be proved directly from the definition of derivative, using the two fundamental limits $\lim_{h\to 0}\frac{\sin h}{h}=1$ and $\lim_{h\to 0}\frac{\cos h-1}{h}=0$. Point out the important role of the angle sum formulas when deriving both derivatives. • The rules for differentiating the other four basic trigonometric functions can be shown easily by using appropriate identities and the rules for differentiating the sine and cosine functions. • Students sometimes forget or misapply the basic trigonometric identities. It may be a good idea to review the reciprocal, quotient, Pythagorean, and angle sum identities. • Remind students that variables in trigonometric functions are always measured in radians. **Possible Assessment Strategies:** • Find the derivative of each of the following functions. a. $f(x)=3x^2-2\cos x$ b. $f(x)=\frac{x}{2-\tan x}$ c. $f(x)=x^2\cos x$ d. $f(x)=\frac{x\sin x}{1+x}$ • Find an equation of the tangent line to the curve $y=2x\sin x$ at the point where $x=\frac{\pi}{2}$.

UNIT 3
DERIVATIVES II

SUGGESTED TIME
15 classes

Section 4.1 – Chain Rule (pp. 153-161)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C9 (A B C)** **After this lesson, students will be expected to:** • differentiate composite functions using the Chain Rule **After this lesson, students should understand the following concepts:** • **Chain Rule –** a rule used to differentiate composite functions; if two functions *f* and *g* are both differentiable, and $F(x)=(f\circ g)(x)=f[g(x)]$, then $$F'(x)=(f\circ g)'(x)=f'[g(x)]\bullet g'(x)$$ • **Power Chain Rule –** if $f(u)=u^n$, where *u* is a function of *x*, then $$\frac{d}{dx}u^n=nu^{n-1}\frac{du}{dx}$$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 158-161:** #1-40 **Suggested Problems in *Calculus* [7E]:** • **pp. 154-156:** #1-54	**Possible Instructional Strategies:** • The text begins by taking a traditional approach for teaching the correct usage of the Chain Rule. First, students are taught to differentiate $y=f[g(x)]$ by setting $u=g(x)$, calculating the two derivatives $f'(u)$ and $g'(x)$, and then applying the Chain Rule to obtain $y'=f'(u)g'(x)=f'[g(x)]g'(x)$. This process is then shortened by dispensing with the *u* and simply referring to $g(x)$ as the inside function. This abbreviated process is called the Outside-Inside Rule. Students should get plenty of practice with the Chain Rule so that its use becomes automatic. • When presenting the Leibniz form of the Chain Rule, $\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$, emphasize that $\frac{dy}{du}$ is evaluated at $u=g(x)$ and $\frac{du}{dx}$ is evaluated at *x*. • In applying the Outside-Inside Rule to differentiate $f[g(x)]$, a common mistake is to omit $g'(x)$ in the answer. **Possible Assessment Strategies:** • Find the derivative of each of the following functions. a. $f(x)=\left(x^4+3x^2-2\right)^5$ b. $f(x)=(2x-3)^4\left(x^2+x+1\right)^5$ c. $f(x)=\left(\frac{x^2+1}{x^2-1}\right)^3$ d. $f(x)=\frac{x+1}{\sin^2 x}$ • Find the value of $(f\circ g)'(x)$ for each of the following functions at the indicated point. a. $f(x)=u^{10},\ u=g(x)=1+2x,\ x=0$ b. $f(x)=\sin u,\ u=g(x)=\cos x,\ x=\frac{\pi}{2}$

Section 4.2 – Implicit Differentiation (pp. 162-169)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C9 (D E F G)** **After this lesson, students will be expected to:** • find derivatives, using implicit differentiation • find derivatives, using the Power Rule for Rational Powers of *x* **After this lesson, students should understand the following concept:** • **implicit differentiation –** a method of differentiating a function that is defined implicitly, without needing to solve the original equation for *y* in terms of *x* **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 167-169:** #1-47 **Suggested Problems in *Calculus* [7E]:** • **pp. 161-163:** #1-32	**Possible Instructional Strategies:** • Explain to students that implicit differentiation is often used to derive rules for differentiating inverse functions. • It is easy to make mistakes in taking derivatives which require the Product Rule and/or the Chain Rule. For example, students may forget to use the Product Rule when differentiating the term *xy*. **Possible Assessment Strategies:** • For each of the following relations, find $\frac{dy}{dx}$ by implicit differentiation. a. $9x^2 - y^2 = 1$ b. $x^2 + xy - y^2 = 4$ c. $y \cos x = x^2 + y^2$ d. $y^5 + x^2y^3 = 1 + x^4y$ • For each of the following relations, find the slope of the curve at the given point. a. $x^2 + xy + y^2 = 3, \quad (1,1)$ b. $x^{1/3} + y^{1/3} = 3, \quad (8,1)$ • Determine the point(s) on the graph of $x^2 + 2xy - y^2 = 9$ where the slope of the curve is not defined. • Find the equation of the tangent and the normal lines to the curve $xy + y^3 = 3$ at the point $(2,1)$. • Use implicit differentiation to find $\frac{d^2y}{dx^2}$ for the relation $9x^2 - 4y^2 = 36$. • Find the derivative of each of the following functions. a. $f(x) = \sqrt{3x^2 - 1}$ b. $f(x) = (\sin x)^{-2/3}$ c. $f(x) = \frac{\sqrt{x}}{3x - 5}$

Section 1.6 – Trigonometric Functions (pp. 45-53)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C10 (A B C D E F)** **After this lesson, students will be expected to:** • use the inverse trigonometric functions to solve problems **After this lesson, students should understand the following concepts:** • **inverse cosine function –** the inverse of the cosine function with restricted domain $[-1,1]$ • **inverse sine function –** the inverse of the sine function with restricted domain $[-1,1]$ • **inverse tangent function –** the inverse of the tangent function with restricted domain $(-\infty,\infty)$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 51-53:** #27-42 **Suggested Problems in *Calculus* [7E]:** • **pp. 459-461:** #1-16	**Possible Instructional Strategies:** • Ensure that students are comfortable working with inverse trigonometric functions. • Remind students that $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$, but instead means the inverse of $\sin x$. **Possible Assessment Strategies:** • Solve each of the following equations in the specified interval. Round off all answers to four decimal places. a. $\cos x = 0.4,\ 0 \le x < \pi$ b. $\sin x = -\frac{3}{4},\ -\frac{\pi}{2} \le x < \frac{\pi}{2}$ c. $\tan x = 2,\ -\frac{\pi}{2} \le x < \frac{\pi}{2}$ • Find the exact value of each expression. a. $\cos^{-1}(0.5)$ b. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ c. $\tan^{-1}(-\sqrt{3})$ d. $\cos\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$ e. $\tan\left[\sin^{-1}\left(\frac{2}{5}\right)\right]$ • Simplify each of the following expressions. a. $\cos(\sin^{-1} x)$ b. $\tan(\cos^{-1} x)$

Section 4.3 – Derivatives of Inverse Trigonometric Functions (pp. 170-176)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C10 (G H)** **After this lesson, students will be expected to:** • calculate derivatives of functions involving the inverse trigonometric functions **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 175-176:** #1-29 **Suggested Problems in *Calculus* [7E]:** • **pp. 459-461:** #17-33	**Possible Instructional Strategies:** • Students should understand how the derivatives of inverse functions are derived. An understanding of these concepts is much more useful than memorizing formulas. • The inverse function-inverse cofunction identities and calculator conversion identities are very important. It is essential for students to know how to enter all inverse trigonometric functions into a calculator. **Possible Assessment Strategies:** • Find the derivative of each of the following functions. a. $y=\tan^{-1}\left(x^2\right)$ b. $y=\tan^{-1}\left(\cos x\right)$ c. $y=x\sin^{-1}x+\sqrt{1-x^2}$ d. $y=\sin^{-1}\left(2x+1\right)$ • Find the equation of the tangent line to the curve $y=\cos^{-1}x$ at the point where $x=\frac{1}{2}$.

Section 4.4 – Derivatives of Exponential and Logarithmic Functions (pp. 177-185)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C11 (A B C D E F G)** **After this lesson, students will be expected to:** • calculate derivatives of exponential and logarithmic functions **After this lesson, students should understand the following concept:** • **logarithmic differentiation –** the process of taking the natural logarithm of both sides of a relation, differentiating, and then solving for the desired derivative **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 183-185:** #1-53 **Suggested Problems in *Calculus* [7E]:** • **pp. 401-403:** #31-52 • **pp. 418-421:** #1-38, 43-54	**Possible Instructional Strategies:** • Have a discussion regarding the graph of $y = e^x$ and having students decide what properties the derivative of this function ought to have. • A brief review of the properties of logarithms is appropriate, since students will need these concepts in order to understand the material in this section. • Ensure that students understand why the derivative of $y = x^x$ cannot be found directly using the Power Rule. **Possible Assessment Strategies:** • Find the derivative of each of the following functions. a. $f(x) = 5e^{2x}$ b. $f(x) = x^2e^{x-2}$ c. $f(x) = (x^3 + 2x)e^x$ d. $f(x) = e^x \sin x$ e. $f(x) = \ln(2x+3)$ f. $f(x) = x \ln x - 3x^2$ g. $f(x) = 7^{3x^2}$ h. $f(x) = \log_5(3x-1)$ • Find the equation of the tangent line to the curve $y = e^{5x}$ at the point where $x = 0$. • Use logarithmic differentiation to find the derivative of each of the following functions. a. $f(x) = x^{\sin x}$ b. $f(x) = \frac{(3x+1)^4(x-1)^6}{\sqrt{6x-5}}$

UNIT 4
APPLICATIONS OF DERIVATIVES

SUGGESTED TIME

20 classes

Section 5.1 – Extreme Values of Functions (pp. 191-199)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C12 (A B C)** **After this lesson, students will be expected to:** • determine the local and absolute extreme values of a function **After this lesson, students should understand the following concepts:** • **absolute maximum value –** a function *f* has an absolute maximum value $f(c)$ at a point *c* in its domain *D* if and only if $f(x) \le f(c)$ for all *x* in *D*; also called a global maximum value • **absolute minimum value –** a function *f* has an absolute minimum value $f(c)$ at a point *c* in its domain *D* if and only if $f(x) \ge f(c)$ for all *x* in *D*; also called a global minimum value • **absolute extrema –** absolute maximum and absolute minimum values of a function • **local maximum value –** a function *f* has a local maximum value $f(c)$ at a point *c* in the interior of its domain if and only if $f(x) \le f(c)$ for all *x* in some open interval containing *c*; also called a relative maximum value • **local minimum value –** a function *f* has a local minimum value $f(c)$ at a point *c* in the interior of its domain if and only if $f(x) \ge f(c)$ for all *x* in some open interval containing *c*; also called a relative minimum value • **local extrema –** local maximum and local minimum values of a function • **critical point –** a point *c* in the domain of *f* at which $f'(c) = 0$ or $f'(c)$ does not exist • **stationary point –** a point *c* in the domain of *f* at which $f'(c) = 0$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 198-199:** #1-30 **Suggested Problems in *Calculus* [7E]:** • **pp. 204-206:** #1-56	**Possible Instructional Strategies:** • Sketch an arbitrary function and discuss the local and absolute maxima and minima of the function. Students can understand the definitions of these concepts more easily if they have an intuitive introduction to their meaning. • An understanding of minima and maxima is critical to the study of the applications of a derivative. Students need to understand the language of calculus, so emphasize the terminology of this section. Since conformation of what is seen comes from analysis, it is important to incorporate previously acquired algebraic skills into the study of calculus. • Some students will assume that a critical point always corresponds to a local extreme value. It is essential that students see some examples when this is not the case. • When finding the critical points of a function, some students will neglect to consider the points where the derivative is undefined. **Possible Assessment Strategies:** • For each of the following functions, use analytic methods to find the absolute and local extreme values of the function on the given interval. a. $f(x) = 1 + (x+1)^2, \quad -2 \le x \le 5$ b. $f(x) = \frac{1}{x}, \quad x \ge 1$ c. $f(x) = 3x - x^3, \quad -3 \le x \le 3$ d. $f(x) = \cos x, \quad -\pi \le x \le \pi$ • Find all absolute and local extreme values of each of the following functions. a. $f(x) = 4 + 6x - 3x^2$ b. $f(x) = \frac{x-1}{x^2 - x + 1}$ c. $y = x^3 + 3x^2 - 1$

Section 5.2 – Mean Value Theorem (pp. 200-208)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C12 (D E)** **After this lesson, students will be expected to:** • apply the Mean Value Theorem and find the intervals on which a function is increasing or decreasing **After this lesson, students should understand the following concepts:** • **increasing function –** if *f* is a function defined on an open interval *I*, then *f* increases on *I* if, for any two points x_1 and x_2, $x_1 < x_2$ implies that $f(x_1) < f(x_2)$ • **decreasing function –** if *f* is a function defined on an open interval *I*, then *f* decreases on *I* if, for any two points x_1 and x_2, $x_1 < x_2$ implies that $f(x_1) > f(x_2)$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 206-208:** #1-28 **Suggested Problems in *Calculus* [7E]:** • **pp. 220-223:** #9-14 (a-b), #29-40 (a-b)	**Possible Instructional Strategies:** • Please note that this resource uses closed intervals to describe intervals on which a function increases and decreases, whereas most other resources use open intervals. In order to be consistent with the vast majority of university resources, this curriculum guide will use open intervals to describe increasing and decreasing intervals. **Possible Assessment Strategies:** • For each of the following functions, use analytic methods to find ➢ the local extrema, ➢ the intervals on which the function is increasing, ➢ the intervals on which the function is decreasing. a. $f(x) = 2x^3 + 3x^2 - 36x$ b. $f(x) = x^4 - 2x^2 + 3$ c. $f(x) = \frac{x}{x^2+1}$ d. $f(x) = x\sqrt{x-9}$ • Sketch a graph of a differentiable function $y = f(x)$ that has a local maximum at $(0,1)$ and absolute minima at $(-1,0)$ and $(1,0)$.

Section 5.3 – Connecting f' and f'' with the Graph of f (pp. 209-222)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C12 (F G H I J)** **After this lesson, students will be expected to:** • use the First and Second Derivative Tests to determine the local extreme values of a function • determine the concavity of a function and locate the points of inflection by analysing the second derivative • graph *f* using information about f' **After this lesson, students should understand the following concepts:** • **concave up –** the graph of a function *f* is said to be concave up on an interval (a,b) if it lies above all of its tangents in (a,b); a graph is concave up if f' is increasing on (a,b) • **concave down –** the graph of a function *f* is said to be concave down on an interval (a,b) if it lies below all of its tangents in (a,b); a graph is concave down if f' is decreasing on (a,b) • **point of inflection –** a point where the graph of a function has a tangent line and the concavity changes **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 219-222:** #1-30, 33-40 **Suggested Problems in *Calculus* [7E]:** • **pp. 220-223:** #9-14 (c), #29-40 (c) • **pp. 242-244:** #1-10	**Possible Instructional Strategies:** • Present several graphs of functions to the students, have them discuss their derivatives and explain how they relate to the extreme values of each function. • It is vital that students understand the connection between the graph of a function, and the graphs of its first and second derivatives. The analysis of the first and second derivatives establishes all the important features suggested by the graph on a graphing calculator. • Students will need to learn to use their own judgment in deciding which test to apply in finding the local extreme values of a function. Sometimes y'' is too complicated or lengthy to find algebraically, so the First Derivative Test may be easier to use than the Second Derivative Test. • If is important for students to understand that the condition $f'(c)=0$ does not guarantee that *f* has a local extremum at $(c, f(c))$. Likewise, some students will identify any points for which $f''(x)=0$ as points of inflection. Remind them that a change in concavity must exist in order for a function to have a point of inflection. **Possible Assessment Strategies:** • For each of the following functions, use analytic methods to find ➢ the points of inflection, ➢ the intervals on which the function is concave up, ➢ the intervals on which the function is concave down. a. $f(x)=8x-x^3$ b. $f(x)=4x^3-3x^2-6x+1$ • Use analytic methods to sketch the graph of each of the following functions. a. $f(x)=2+3x^2-x^3$ b. $f(x)=\frac{x}{x^2-9}$ c. $f(x)=(x-3)\sqrt{x}$

Section 5.4 – Modeling and Optimization (pp. 223-236)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C13 (A B C D)** **After this lesson, students will be expected to:** • solve application problems involving finding minimum or maximum values of a function **After this lesson, students should understand the following concept:** • **optimization –** in an application, maximizing or minimizing some aspect of the system being modelled **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 230-236:** #1-50 **Suggested Problems in *Calculus* [7E]:** • **pp. 256-262:** #1-40	**Possible Instructional Strategies:** • Ask students to suggest situations in which one might want to find the minimum or maximum values of a function. • Students traditionally have difficulty with optimization problems, particularly with the formation of the function to be optimized and the determination of the domain for the problem situation. In this regard, stress the six-step strategy for solving optimization problems with the students. • In optimization problems, students may overlook endpoints as possible candidates for optimal values, or use solutions that are outside of the domain of the input value. **Possible Assessment Strategies:** • The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares? • A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area? • A store has been selling 200 Blu-Ray disc players a week at $350 each. A market survey indicates that for each $10 rebate offered to buyers, the number of units sold will increase by 20 per week. At what price should the store sell its Blu-Ray disc players to maximize its revenue? • Find the point on the parabola $y = \frac{1}{2}x^2$ that is closest to the point $(4,1)$. • A cylindrical can is to be made to hold 1 L of oil, which has a volume of 1000 cm^3. Find the dimensions that will minimize the cost of the metal to manufacture the can. Round off the answers to one decimal place.

Section 5.5 – Linearization and Differentials (pp. 237-249)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C14 (A B C D)** **After this lesson, students will be expected to:** • find linearizations and use Newton's method to approximate the zeros of a function • estimate the change in a function using differentials **After this lesson, students should understand the following concepts:** • **linearization –** the approximating function $L(x) = f(a) + f'(a)(x-a)$ when *f* is differentiable at $x = a$ • **differential –** if $y = f(x)$ is a differentiable function, the differential *dx* is an independent variable and the differential *dy* is $dy = f'(x)dx$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 246-249:** #1-26, 53-56 **Suggested Problems in *Calculus* [7E]:** • **pp. 187-188:** #1-28 • **pp. 267-268:** #1-26	**Possible Instructional Strategies:** • Linearizations can provide useful estimates when dealing with complicated functions, especially if a calculator is not available. • It is important to note the connection between linearization and estimates of change, that is, the differential estimate of the change in *f* when *x* changes by *dx* is based on a linearization of *f*. • Newton's Method is a fast, efficient way to approximate roots of differentiable functions. • When using Newton's Method to find the zeros of a function, some students may stop after finding one zero or may not choose appropriate values for the initial guess x_1. **Possible Assessment Strategies:** • For each of the following functions, find the linearization at the given point, a. $f(x) = \sqrt{x+3},\ a = 1$ b. $f(x) = \cos x,\ a = \frac{\pi}{4}$ • Approximate each root by using a linearization centered at an appropriate nearby number. Round off all answers to four decimal places. a. $\sqrt{120}$ b. $\sqrt[3]{126}$ • For each of the following ➢ find *dy*, ➢ evaluate *dy* for the given values of *x* and *dx*, to four decimal places. a. $y = x^2 - 4x,\ x = 3,\ dx = 0.1$ b. $y = \frac{x}{x+1},\ x = 1,\ dx = 0.05$ • Use Newton's Method to determine all real solutions of each equation. Round off all answers to four decimal places. a. $x^3 - 2x - 5 = 0$ b. $\cos x = x$

Section 5.6 – Related Rates (pp. 250-259)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C15 (A B C)** **After this lesson, students will be expected to:** • solve related rate problems **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 255-259:** #1-35 **Suggested Problems in *Calculus* [7E]:** • **pp. 180-183:** #1-20	**Possible Instructional Strategies:** • Review the Chain Rule and implicit differentiation with the class. A mastery of these topics is crucial for success in solving related rate problems where there is not a direct functional relationship between two quantities. • Go over and model the six-step strategy for solving related rate problems with the students. • The most common student error in solving related rate problems is substituting a value for a variable too early, which makes it impossible to take the appropriate derivatives. Emphasize that evaluation is the final step in solving a related rates problem. **Possible Assessment Strategies:** • Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm? • A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall? • Car A is travelling west at 50 mph and Car B is travelling north at 60 mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when Car A is 0.3 miles and Car B is 0.4 miles from the intersection? • A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. • A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

UNIT 5
INTEGRALS AND THEIR APPLICATIONS

SUGGESTED TIME
18 classes

Section 6.1 – Estimating with Finite Sums (pp. 267-277)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C17 (A)** **After this lesson, students will be expected to:** • approximate the area under the graph of a non-negative continuous function by using rectangle approximation methods • interpret the area under a graph as a net accumulation of a rate of change **After this lesson, students should understand the following concepts:** • **integral calculus –** the branch of mathematics that deals with integrals • **LRAM –** left-hand endpoint rectangular approximation method; the method of approximating a definite integral over an interval, using the function values at the left-hand endpoints of the subintervals determined by a partition • **MRAM –** midpoint rectangular approximation method; the method of approximating a definite integral over an interval, using the function values at the midpoints of the subintervals determined by a partition • **RRAM –** right-hand endpoint rectangular approximation method; the method of approximating a definite integral over an interval, using the function values at the right-hand endpoints of the subintervals determined by a partition **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 274-277:** #1-12 **Suggested Problems in *Calculus* [7E]:** • **pp. 293-295:** #1-8	**Possible Instructional Strategies:** • Several methods of using rectangles to estimate the area under the graph of a non-negative continuous function are introduced in this section. It is important that students first sketch the curve over the desired interval in order to visualize the area being sought. • As *n* increases, students will see the approaching sums converge to a limit, which is fundamental to the idea of integral calculus. An understanding of this can lead to a greater appreciation of the Fundamental Theorem of Calculus • Some students may assume that the MRAM estimate will always be the average of the LRAM and RRAM estimates. Give an example to show that this is not always the case. **Possible Assessment Strategies:** • For each of the following, use MRAM to estimate the area of the region enclosed between the graph of *f* and the *x*-axis for $a \le x \le b$ with the given value of *n*. Round off the answers to four decimal places, where necessary. a. $y = x^2 + 3x,\ a = 0,\ b = 2,\ n = 10$ b. $y = \frac{1}{x},\ a = 1,\ b = 3,\ n = 10$ c. $y = \sin x,\ a = 0,\ b = \pi,\ n = 8$

Section 6.2 – Definite Integrals (pp. 278-288)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C17 (B C)** **After this lesson, students will be expected to:** • express the area under a curve as a definite integral and as a limit of Riemann sums • compute the area under a curve using a numerical integration procedure **After this lesson, students should understand the following concepts:** • **partition –** a set of numbered points in $[a,b]$, ordered from left to right such that the first point is *a* and the last point is *b* • **subinterval –** a part of the interval $[a,b]$, such that the left and right side of the subinterval are consecutive points of a partition • **Riemann sum –** a sum of the form $$\sum_{k=1}^{n} f(c_k)\bullet\Delta x_k$$ where *f* is a continuous function on a closed interval $[a,b]$, Δx_k is the length of the *k*th subinterval in some partition of $[a,b]$, and c_k is a point in the subinterval • **norm of a partition –** the longest subinterval length, denoted $\lVert P\rVert$, for partition *P* • **integrable function –** a function for which the definite integral over a closed interval exists • **definite integral –** integrating a function over a closed interval • **regular partition –** a partition in which successive points are regularly spaced • **limits of integration –** the values of *a* and *b* in the expression $\int_a^b f(x)\,dx$ • **integrand –** the expression $f(x)$ in $\int f(x)\,dx$ or $\int_a^b f(x)\,dx$ • **dummy variable of integration –** in $\int_a^b f(x)\,dx$, the variable *x*; it could be any other variable without changing the value of the integral **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 286-288:** #1-22 **Suggested Problems in *Calculus* [7E]:** • **pp. 306-309:** #35-40	**Possible Instructional Strategies:** • Some students may not have seen sigma notation before. As a result, it may be necessary to spend some time discussing this notation with the class, as it is essential to the understanding of this lesson. • The formal definition of the definite integral as a limiting value of Riemann sums is presented in this section. Emphasize that LRAM, MRAM, and RRAM are examples of Riemann sums. • In writing definite integrals, students will often omit the differential, *dx*. Remind them that the differential is a required part of an integral expression. **Possible Assessment Strategies:** • Express each of the following limits as a definite integral. a. $\lim_{n\to\infty}\sum_{k=1}^{n}\left(c_k^{\,2}+3\right)\Delta x_k,\ \ [2,4]$ b. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{c_k+1}\Delta x_k,\ \ [0,3]$ • Evaluate each of the following integrals. a. $\int_1^6 7\,dx$ b. $\int_{-1}^3 4\,dx$ • Use the graph of the integrand and areas to evaluate each of the following integrals. a. $\int_2^5 (2x-1)\,dx$ b. $\int_{-2}^2 \sqrt{4-x^2}\,dx$ c. $\int_{-2}^5 \lvert 2+x\rvert\,dx$

Section 6.3.1 – Definite Integrals and Antiderivatives (pp. 289-297)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C17 (D)** **After this lesson, students will be expected to:** • apply the rules for definite integrals **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 294-297:** #1-6	**Possible Instructional Strategies:** • Ensure that students understand and are able to apply the rules for definite integrals. **Possible Assessment Strategies:** • Suppose that *f* and *g* are continuous and that $\int_1^2 f(x)dx = 3$, $\int_1^5 f(x)dx = 7$, and $\int_1^5 g(x)dx = 8$. Determine the value of each definite integral. a. $\int_2^1 f(x)dx$ b. $\int_5^5 g(x)dx$ c. $3\int_1^5 g(x)dx$ d. $\int_1^5 f(x)dx + \int_1^5 g(x)dx$

Section 6.3.2 – Definite Integrals and Antiderivatives (pp. 289-297)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C16 (A B C)** **After this lesson, students will be expected to:** • determine the antiderivative of a function **After this lesson, students should understand the following concepts:** • **antiderivative –** the family of functions $f(x)+C$, where *C* is a constant, with common derivative $f'(x)$ • **antidifferentiation –** the process of going from a derivative function to a function that has that derivative • **indefinite integral of a function *f* –** the set of all antiderivatives of *f*, denoted by $\int f(x)\,dx$ • **constant of integration –** the arbitrary constant *C* in $\int f(x)\,dx = F(x)+C$, where *F* is any antiderivative of *f* **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 206-208:** #29-38 **Suggested Problems in *Calculus* [7E]:** • **pp. 273-275:** #1-40 • **pp. 326-329:** #5-12, 19-24	**Possible Instructional Strategies:** • The concept of antiderivative is a very important one in the study of calculus. Have students discuss strategies for finding antiderivatives of simple functions, such as those presented in this section. • Students often make mistakes when finding antiderivatives. They should get into the habit of differentiating their answer to verify that they have found the correct antiderivative. **Possible Assessment Strategies:** • For each the following, find all possible functions with the given derivative. a. $f'(x) = x-3$ b. $f'(x) = \frac{1}{2}+\frac{3}{4}x^2-\frac{4}{5}x^3$ c. $f'(x) = (x+1)(2x-1)$ d. $f'(x) = 2\cos x$ e. $f'(x) = \frac{5}{x}$ • For each the following, find the function that satisfies the given conditions. a. $f'(x) = 2x-3,\ f(0)=3$ b. $f'(x) = \frac{1}{2}x^2-2x+6,\ f(0)=-1$ • Find each of the following indefinite integrals. a. $\int (x^2+x^{-2})\,dx$ b. $\int \frac{x^4-2\sqrt{x}}{x}\,dx$ c. $\int (x-3)(2x+1)\,dx$

Section 6.4 – Fundamental Theorem of Calculus (pp. 298-309)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C17 (E F)** **After this lesson, students will be expected to:** • apply the Fundamental Theorem of Calculus • understand the relationship between the derivative and definite integral as expressed in both parts of the Fundamental Theorem of Calculus **FUNDAMENTAL THEOREM OF CALCULUS** If *f* is continuous on $[a,b]$, then the function $F(x)=\int_a^x f(t)\,dt$ has a derivative with respect to *x* at every point in $[a,b]$, and $$\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$ If *f* is continuous on $[a,b]$, and *F* is any antiderivative of *f* on $[a,b]$, then $$\int_a^b f(x)\,dx=F(b)-F(a)$$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 306-309:** #1-8, 27-44 **Suggested Problems in *Calculus* [7E]:** • **pp. 318-320:** #19-30	**Possible Instructional Strategies:** • Remind students that the Fundamental Theorem of calculus shows the connection between differential calculus and integral calculus. **Possible Assessment Strategies:** • For each of the following, find $\frac{dy}{dx}$. a. $y=\int_0^x \cos^2 t\ dt$ b. $y=\int_6^x \left(t^2-3t+1\right)dt$ c. $y=\int_x^3 e^{3t}\ dt$ • Evaluate each of the following definite integrals. a. $\int_{-2}^3 \left(3-x^2\right)dx$ b. $\int_1^4 \left(\frac{4}{x^3}-\frac{1}{x^2}\right)dx$ c. $\int_{-1}^2 \left(x^3-2x\right)dx$ d. $\int_1^4 \left(5-2x+3x^2\right)dx$ e. $\int_1^9 \sqrt{x}\ dx$ f. $\int_{-1}^2 (x-2)(x+3)\,dx$ g. $\int_{\pi/4}^{\pi} \sin\ x\ dx$ h. $\int_0^1 e^x dx$

Section 7.2 – Antidifferentiation By Substitution (pp. 336-344)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C16 (D)** **After this lesson, students will be expected to:** • determine indefinite integrals and calculate definite integrals using the method of substitution **After this lesson, students should understand the following concept:** • **antidifferentiation by substitution –** a method of integration in which $\int f\left[g\left(x\right)\right] \bullet g'\left(x\right)\,dx$ is rewritten as $\int f\left(u\right)\,du$ by substituting $u = g\left(x\right)$ and $du = g'\left(x\right)dx$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 342-344:** #1-24, 53-66 **Suggested Problems in *Calculus* [7E]:** • **pp. 335-337:** #1-4, 7-11	**Possible Instructional Strategies:** • A review of the Chain Rule is an effective way to begin this section, since a *u*-substitution is a method for reversing the process of using the Chain Rule. • One of the most common mistakes made by students is to insert the wrong constant multiplier when using the substitution method. To prevent this type of mistake, emphasize the mechanical nature of the process, once the correct substitution is identified. For example, if $u = 2x$, then $du = 2\,dx$, so we may solve for *dx* to obtain $dx = \frac{1}{2}du$. Then we simply need to make the appropriate substitution for *dx* in the integral problem. • Many different types of mistakes occur when substituting into definite integrals. Students who have difficulty should be encouraged to write the variable name with the limits of integration in order to avoid mistakes, such as writing $\int_{u=3}^{u=5} u^2\,du$. **Possible Assessment Strategies:** • Find each of the following indefinite integrals, using an appropriate substitution. a. $\int 2x\sqrt{1+x^2}\,dx$ b. $\int x^3\cos\left(x^4+2\right)dx$ • Evaluate each of the following definite integrals, using an appropriate substitution. a. $\int_0^4 \sqrt{2x+1}\,dx$ b. $\int_1^2 \frac{dx}{\left(3-5x\right)^2}$

Section 8.2 – Areas in the Plane (pp. 394-402)

ELABORATIONS & SUGGESTED PROBLEMS	POSSIBLE INSTRUCTIONAL & ASSESSMENT STRATEGIES
Specific Curriculum Outcome(s) and Achievement Indicator(s) addressed: • **C18 (A B)** **After this lesson, students will be expected to:** • use integration to calculate areas of regions in a plane **After this lesson, students should understand the following concept:** • **area between curves –** if *f* and *g* are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then the area between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is $$A = \int_a^b [f(x) - g(x)]dx$$ **Suggested Problems in *Calculus: Graphical, Numerical, Algebraic*:** • **pp. 306-309:** #41-44 • **pp. 399-402:** #1-34 **Suggested Problems in *Calculus* [7E]:** • **pp. 349-350:** #1-2, 5-10, 13-16	**Possible Instructional Strategies:** • The first step in finding the area of a region between two curves is to graph the region. By graphing the two curves, students will be able to see which functions define the upper and lower boundaries of the region. • Determining the limits of integration may involve finding the points where $y = f(x)$ and $y = g(x)$ intersect. This will mean solving the equation $f(x) = g(x)$ to find the *x*-coordinates of the intersection points. They will become the limits of the definite integral. **Possible Assessment Strategies:** • For each of the following, find the area of the region between the curve and the *x*-axis. a. $y = 2 + x,\ 1 \leq x \leq 4$ b. $y = 2x^2 - 1,\ 1 \leq x \leq 3$ c. $y = x^3 - 1,\ 2 \leq x \leq 3$ d. $y = x^3 - 4x,\ 2 \leq x \leq 3$ • Find the area of the region enclosed by each group of equations. a. $y = 9 - x^2,\ y = x + 1,\ x = -1,\ x = 2$ b. $y = x,\ y = \sin x,\ x = \frac{\pi}{2},\ x = \pi$ c. $y = (x - 2)^2,\ y = x$ d. $y = x^2 - 2x,\ y = x + 4$

GLOSSARY OF MATHEMATICAL TERMS

A

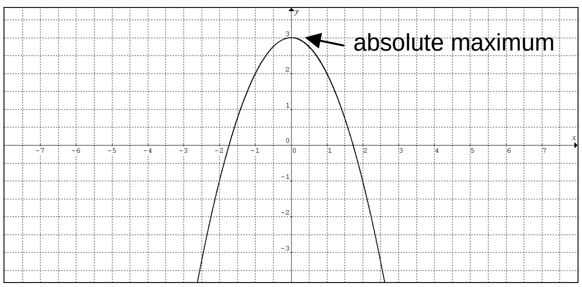
- **absolute change –** as we move from $x = a$ to a nearby point $x = a + dx$, the absolute change is defined as

$$\Delta f = f(a + dx) - f(a)$$

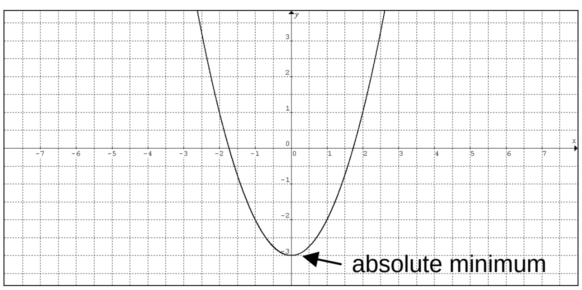
- **absolute error –** the absolute value of the difference between the true value and the approximate value

$$\text{absolute error} = |\text{true value} - \text{approximate value}|$$

- **absolute extrema –** absolute maximum and absolute minimum values of a function
- **absolute maximum value –** a function *f* has an absolute maximum value $f(c)$ at a point *c* in its domain *D* if and only if $f(x) \le f(c)$ for all *x* in *D*; also called a global maximum value



- **absolute minimum value –** a function *f* has an absolute minimum value $f(c)$ at a point *c* in its domain *D* if and only if $f(x) \ge f(c)$ for all *x* in *D*; also called a global minimum value

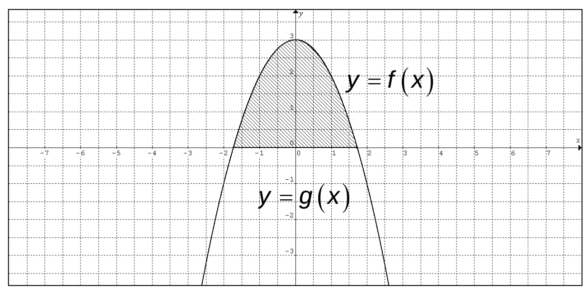


- **acceleration –** the rate of change of the velocity of an object with respect to time

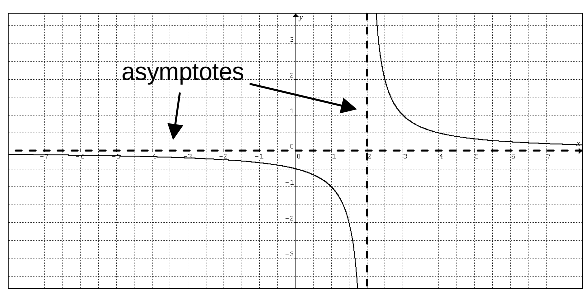
$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

- **antiderivative –** the family of functions $f(x) + C$, where *C* is a constant, with common derivative $f'(x)$
- **antidifferentiation –** the process of going from a derivative function to a function that has that derivative
- **antidifferentiation by substitution –** a method of integration in which $\int f[g(x)] \cdot g'(x)\,dx$ is rewritten as $\int f(u)\,du$ by substituting $u = g(x)$ and $du = g'(x)dx$
- **arbitrary constant –** see **constant of integration**
- **arccosine function –** see **inverse cosine function**
- **arcsine function –** see **inverse sine function**
- **arctangent function –** see **inverse tangent function**
- **area between curves –** if *f* and *g* are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the area between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is

$$A = \int_a^b [f(x) - g(x)]dx$$



- **asymptote –** a line whose distance to a given curve tends to zero



- **average rate of change –** for a function $y = f(x)$, the average rate of change of *y* with respect to *x* over the interval $x \in [a,b]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

- **average speed –** the distance travelled by an object divided by the elapsed time
- **average velocity –** displacement divided by time travelled; average velocity is calculated using the formula

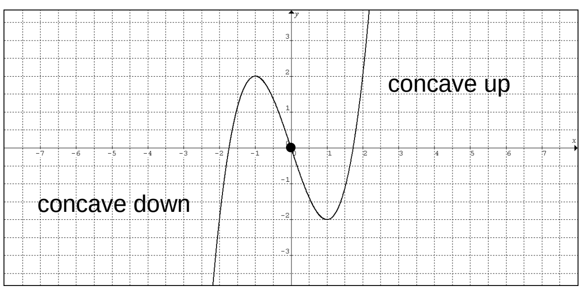
$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

C

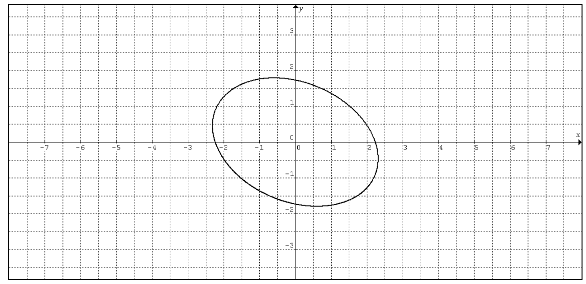
- **Chain Rule –** a rule used to differentiate composite functions; if two functions *f* and *g* are both differentiable, and $F(x) = (f \circ g)(x) = f[g(x)]$, then

$$F'(x) = (f \circ g)'(x) = f'[g(x)] \bullet g'(x)$$

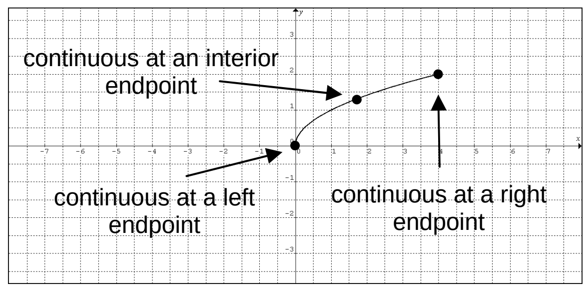
- **concave down –** the graph of a function *f* is said to be concave down on an interval (a,b) if it lies below all of its tangents in (a,b); a graph is concave down if f' is decreasing on (a,b)



- **concave up –** the graph of a function *f* is said to be concave up on an interval (a,b) if it lies above all of its tangents in (a,b); a graph is concave up if f' is increasing on (a,b)
- **connected graph –** a graph that can be drawn as a single, unbroken curve



- **constant of integration –** the arbitrary constant *C* in $\int f(x)\,dx = F(x) + C$, where *F* is any antiderivative of *f*
- **continuous at a left endpoint –** a function $f(x)$ is continuous at a left endpoint $x = a$ of its domain if $\lim_{x \to a^+} f(x) = f(a)$

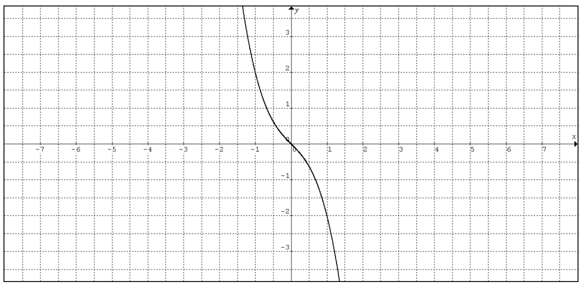


- **continuous at a right endpoint –** a function $f(x)$ is continuous at a right endpoint $x = b$ of its domain if $\lim_{x \to b^-} f(x) = f(b)$
- **continuous at an endpoint –** a function $f(x)$ such that an appropriate one sided limit at an endpoint equals the value of the function at that endpoint
- **continuous at an interior point –** a function $f(x)$ is continuous at an interior point $x = c$ if $\lim_{x \to c} f(x) = f(c)$
- **continuous extension –** a function identical to another function except that is it continuous at one or more points where the other function is not continuous
- **continuous function –** a function that is continuous at every number in its domain
- **continuous on an interval –** a function that is continuous at each point in the interval
- **critical point –** a point *c* in the domain of *f* at which $f'(c) = 0$ or $f'(c)$ does not exist

- **critical value –** see **critical point**

D

- **decreasing function –** if *f* is a function defined on an open interval *I*, then *f* decreases on *I* if, for any two points x_1 and x_2, $x_1 < x_2$ implies that $f(x_1) > f(x_2)$



- **decreasing on an interval –** see **decreasing function**
- **definite integral –** integrating a function over a closed interval
- **derivative of a function at a point *a* –** the value of the derivative of a function $y = f(x)$ at $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  provided the limit exists
- **derivative of a function with respect to *x* –** the derivative of a function $y = f(x)$ is a function defined as

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  provided the limit exists
- **difference quotient –** either of the expressions

$$\frac{f(a+h) - f(a)}{h}$$

  or

$$\frac{f(x) - f(a)}{x - a}$$

- **differentiability –** see **differentiable function**
- **differentiable curve –** the graph of a differentiable function
- **differentiable function –** a function *f* is said to be differentiable at *a* if $f'(a)$ exists; a function that is differentiable at every point of its domain is a differentiable function
- **differentiable on an interval –** a function *f* is said to be differentiable on an interval if it is differentiable at every number in the interval
- **differential –** if $y = f(x)$ is a differentiable function, the differential *dx* is an independent variable and the differential *dy* is $dy = f'(x)dx$
- **differential calculus –** the branch of mathematics that deals with derivatives
- **differentiation –** the process of taking a derivative
- **discontinuity –** see **discontinuous**
- **discontinuous –** a function is discontinuous at $x = a$ if it is not continuous at $x = a$
- **displacement –** the difference between the path of the initial and final position covered by a moving object; displacement is calculated using the formula

$$\Delta s = f(t + \Delta t) - f(t)$$

- **distance travelled –** the integral of the absolute value of a velocity function with respect to time
- **dummy variable of integration –** in $\int_a^b f(x)dx$, the variable *x*; it could be any other variable without changing the value of the integral

E

- **end behaviour model –** the function *g* is an end behaviour model for *f* if it is both a left end and a right end behaviour model for *f*
- **extreme value –** see **extremum**
- **extremum –** a maximum or minimum value (extreme value) of a function on a set

F

- **First Derivative Test (for Local Extrema)**
  - If f' changes from positive to negative at a critical point, then *f* has a local maximum at *c*.
  - If f' changes from negative to positive at a critical point, then *f* has a local minimum at *c*.
  - If f' does not change sign at a critical point *c*, than *f* has no local extreme value at *c*.

- **free fall equation –** when air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls free fall; in a free fall short enough for the acceleration of gravity to be assumed constant, call it *g*, the position of a body released to fall from position s_0 at time $t = 0$ with velocity v_0 is modelled by the equation

$$s(t) = s_0 + v_0 t + \frac{1}{2} g t^2$$

- **Fundamental Theorem of Calculus**
  - **Part 1:** If *f* is continuous on $[a,b]$, then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative with respect to *x* at every point in $[a,b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

  - **Part 2:** If *f* is continuous on $[a,b]$, and *F* is any antiderivative of *f* on $[a,b]$, then

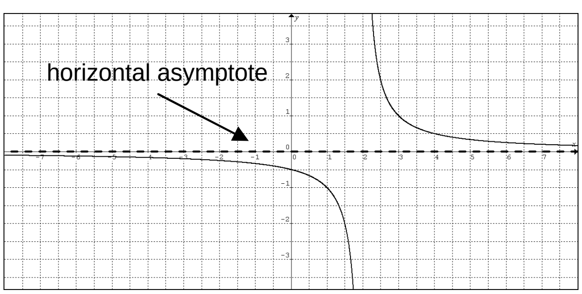
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

G

- **global maximum –** see **absolute maximum**
- **global minimum –** see **absolute minimum**

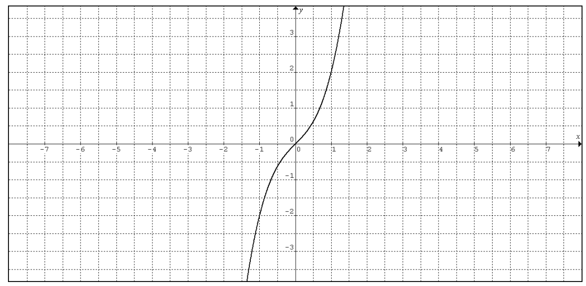
H

- **horizontal asymptote –** the line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, or both

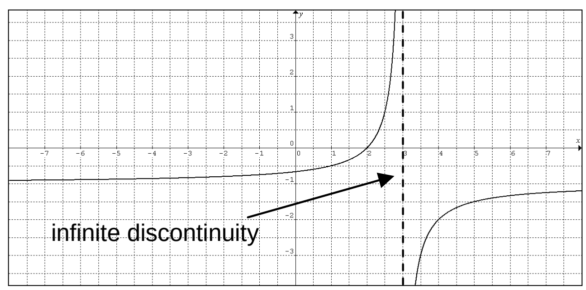


I

- **implicit differentiation –** a method of differentiating a function that is defined implicitly, without needing to solve the original equation for *y* in terms of *x*
- **increasing function –** if *f* is a function defined on an open interval *I*, then *f* increases on *I* if, for any two points x_1 and x_2, $x_1 < x_2$ implies that $f(x_1) < f(x_2)$

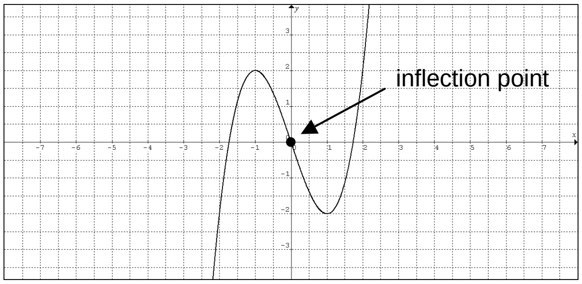


- **increasing on an interval –** see **increasing function**
- **indefinite integral of a function *f* –** the set of all antiderivatives of *f*, denoted by $\int f(x)\,dx$
- **infinite discontinuity –** an infinite discontinuity occurs where a graph has a vertical asymptote



- **infinite limit –** a limit that approaches ∞ or $-\infty$ as *x* approaches a real number; if the values of a function $f(x)$ outgrow all positive bounds as *x* approaches a finite number *a*, we write $\lim_{x\to a} f(x) = \infty$; if the values of a function $f(x)$ become large and negative, exceeding all negative bounds as *x* approaches a finite number *a*, we write $\lim_{x\to a} f(x) = -\infty$

- **inflection point –** a point where the graph of a function has a tangent line and the concavity changes



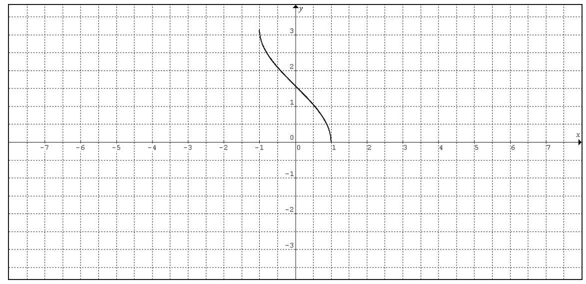
- **instantaneous rate of change –** for a function $y = f(x)$, the instantaneous rate of change of *y* with respect to *x* over the interval $x \in [a, b]$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

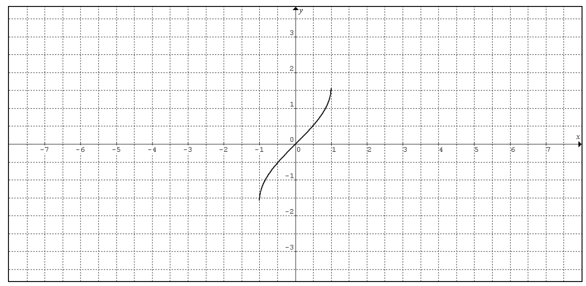
- **instantaneous speed –** the speed of an object at a given instant in time
- **instantaneous velocity –** the derivative of a position function with respect to time; instantaneous velocity is calculated using the formula

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

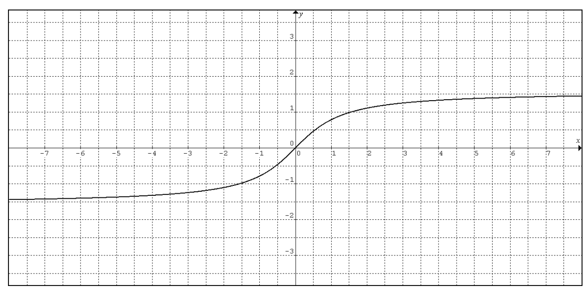
- **integrable function –** a function for which the definite integral over a closed interval exists
- **integral calculus –** the branch of mathematics that deals with integrals
- **integrand –** the expression $f(x)$ in $\int f(x)\,dx$ or $\int_a^b f(x)dx$
- **integration –** the evaluation of a definite integral or an indefinite integral
- **Intermediate Value Property –** the property of a function that has never takes on two values without taking on all values in between those two values
- **Intermediate Value Theorem for Continuous Functions –** a function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every *y*-value between $f(a)$ and $f(b)$
- **Intermediate Value Theorem for Derivatives –** if *a* and *b* are any two points on an interval on which *f* is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$
- **inverse cosine function –** the inverse of the cosine function with restricted domain $[-1, 1]$



- **inverse sine function –** the inverse of the sine function with restricted domain $[-1, 1]$



- **inverse tangent function –** the inverse of the tangent function with domain $(-\infty, \infty)$

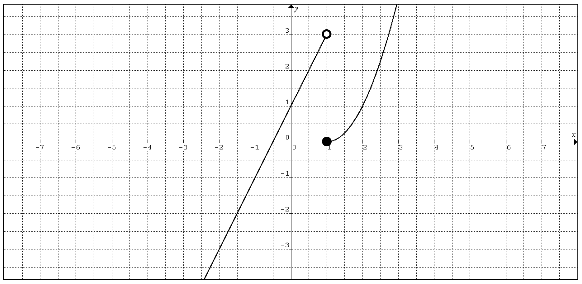


J

- **jerk –** the derivative of an acceleration function with respect to time

$$j(t) = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

- **jump discontinuity –** a function *f* has a jump discontinuity at $x = a$ if $\lim_{x \to a^-} f(x) = L$, $\lim_{x \to a^+} f(x) = M$, and $L \neq M$



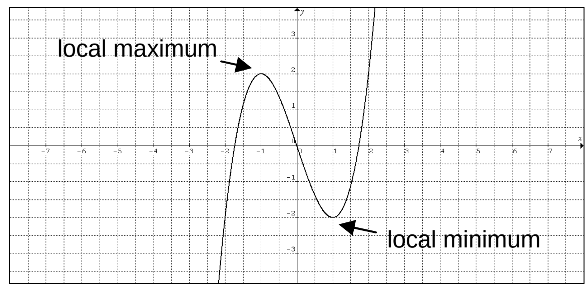
L

- **left end behaviour model –** the function *g* is a left end behaviour model for *f* if and only if $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 1$
- **left-hand derivative –** the derivative defined by a left-hand limit
- **left-hand limit –** a limit, written $\lim_{x \to a^-} f(x)$, which is read "the limit of $f(x)$ as *x* approaches *a* from the left"; used to determine the behaviour of a function, $f(x)$, to the left of $x = a$
- **limit –** a function *f* has a limit *L* as *x* approaches *a*, written $\lim_{x \to a} f(x) = L$, provided that the values of $f(x)$ get closer and closer to *L*, as *x* gets closer and closer to *a*, from both sides of *a*
- **limit at infinity –** a number *L* is a limit at infinity for a function *f* if the values of $f(x)$ get closer and closer to *L* as the absolute values of *x* get larger and larger, with either positive or negative values of *x*; we write $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$, whichever is appropriate
- **limits of integration –** the values of *a* and *b* in the expression $\int_a^b f(x)dx$
- **linear approximation of *f* at *a* –** the approximation $f(x) \approx L(x)$, where $L(x)$ is the linearization of *f* at *a*
- **linearization –** the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

when *f* is differentiable at $x = a$

- **local extrema –** local maximum and local minimum values of a function
- **local linearity –** a feature of all smooth curves; any smooth curve will appear to be linear if you focus on a small enough (local) domain; if a function $f(x)$ is differentiable at $x = a$, then, close to *a*, the graph represents the tangent line at *a*
- **local linearization –** see **linearization**
- **local maximum value –** a function *f* has a local maximum value $f(c)$ at a point *c* in the interior of its domain if and only if $f(x) \leq f(c)$ for all *x* in some open interval containing *c*; also called a relative maximum value



- **local minimum value –** a function *f* has a local minimum value $f(c)$ at a point *c* in the interior of its domain if and only if $f(x) \geq f(c)$ for all *x* in some open interval containing *c*; also called a relative minimum value
- **logarithmic differentiation –** the process of taking the natural logarithm of both sides of a relation, differentiating, and then solving for the desired derivative
- **LRAM –** left-hand endpoint rectangular approximation method; the method of approximating a definite integral over an interval, using the function values at the left-hand endpoints of the subintervals determined by a partition

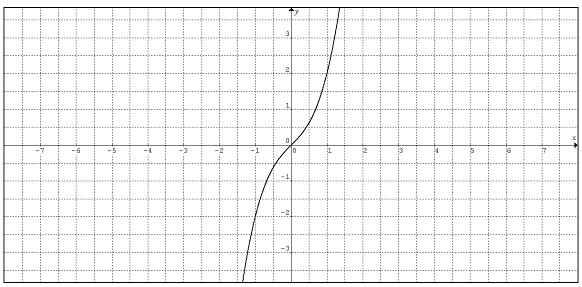
M

- **marginal cost –** the instantaneous rate of change of cost with respect to the number of items produced; the marginal cost is the derivative of the cost function
- **marginal revenue –** the instantaneous rate of change of revenue with respect to the number of items produced; the marginal revenue is the derivative of the revenue function

- **maximum –** see **absolute maximum** and **local maximum**
- **Mean Value Theorem for Derivatives –** if $y = f(x)$ is continuous at every point in the closed interval $[a,b]$ and differentiable at every point in its interior (a,b), then there is at least one point *c* in (a,b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

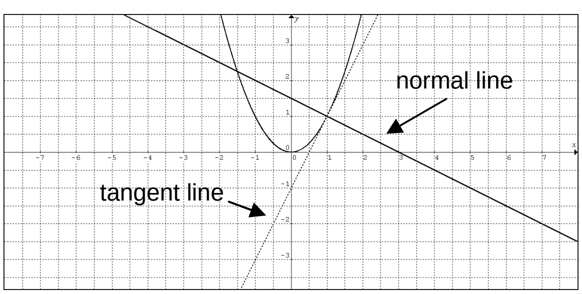
- **minimum –** see **absolute minimum** and **local minimum**
- **monotonic function –** a function that is always increasing on an interval or always decreasing on an interval



- **MRAM –** midpoint rectangular approximation method; the method of approximating a definite integral over an interval, using the function values at the midpoints of the subintervals determined by a partition

N

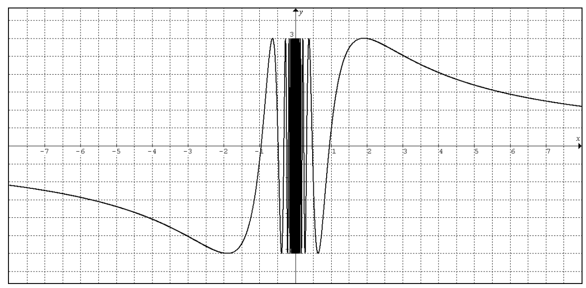
- **nonremovable discontinuity –** a discontinuity that is not removable
- **norm of a partition –** the longest subinterval length, denoted $\|P\|$, for partition *P*
- **normal line to a curve –** the normal line to a curve at a point is the line perpendicular to, and intersecting, the tangent line to the curve at that point



- ***n*th derivative –** an expression indicating that a function is differentiated *n* times; denoted by $\frac{d^n y}{dy^n}$ or $y^{(n)}$
- **numerical method –** a method for generating a numerical solution to a problem; for example, a numerical method for estimating the zeros of an equation is Newton's Method

O

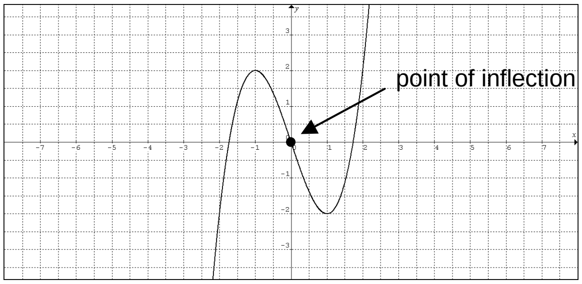
- **one-sided limit –** a left-hand or a right-hand limit
- **optimization –** in an application, maximizing or minimizing some aspect of the system being modelled
- **order of a derivative –** the number of derivatives that are taken of a function
- **oscillating discontinuity –** a point near where the values of a function oscillate too much for the function to have a limit



P

- **partition –** a set of numbered points in $[a,b]$, ordered from left to right such that the first point is *a* and the last point is *b*
- **percentage change –** as we move from $x = a$ to a nearby point $x = a + dx$, the percentage change is defined as $\frac{\Delta f}{f(a)} \cdot 100$
- **point of discontinuity –** a function has a point of discontinuity at $x = a$ if it is not continuous at $x = a$

- **point of inflection –** a point where the graph of a function has a tangent line and the concavity changes



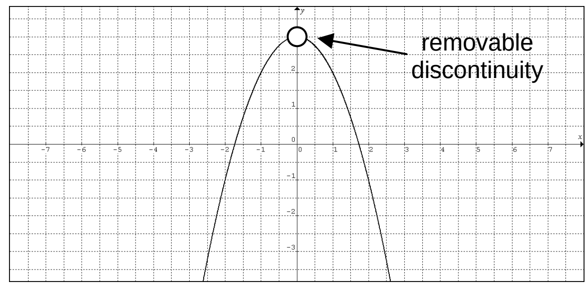
- **position function –** a function *f* that gives the position $f(t)$ of a body on a coordinate axis at time *t*
- **Power Chain Rule –** if $f(u)=u^n$, where *u* is a function of *x*, then
$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$
- **prime notation –** if $y=f(x)$, then both y' and $f'(x)$ denote the derivative of the function with respect to *x*
- **Product Rule –** the product of two differentiable functions *u* and *v* is differentiable, and
$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

Q

- **Quotient Rule –** at a point where $v \neq 0$, the quotient $\frac{u}{v}$ of two differentiable functions *u* and *v* is differentiable, and
$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

R

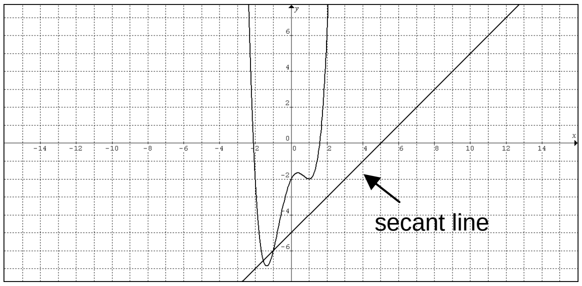
- **rate of change –** see **average rate of change** and **instantaneous rate of change**
- **regular partition –** a partition in which successive points are regularly spaced
- **related rate equation –** an equation involving two or more variables that are differentiable functions with respect to time that can be used to find an equation relating the corresponding rates
- **relative change –** as we move from $x=a$ to a nearby point $x=a+dx$, the relative change is defined as $\frac{\Delta f}{f(a)}$
- **relative extrema –** see **local extrema**
- **relative maximum –** see **local maximum**
- **relative minimum –** see **local minimum**
- **removable discontinuity –** a function *f* has a removable discontinuity at $x=a$ if $\lim_{x\to a} f(x)=L$, and either $f(a)\neq L$ or $f(a)$ does not exist



- **Riemann sum –** a sum of the form
$$\sum_{k=1}^{n} f(c_k)\bullet\Delta x_k$$
where *f* is a continuous function on a closed interval $[a,b]$, Δx_k is the length of the *k*th subinterval in some partition of $[a,b]$, and c_k is a point in the subinterval
- **right end behaviour model –** the function *g* is a right end behaviour model for *f* if and only if
$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$$
- **right-hand derivative –** the derivative defined by a right-hand limit
- **right-hand limit –** a limit, written $\lim_{x\to a^+} f(x)$, which is read "the limit of $f(x)$ as *x* approaches *a* from the right"; used to determine the behaviour of a function, $f(x)$, to the right of $x=a$
- **roundoff error –** error due to rounding numbers that are used in further computations
- **RRAM –** right-hand endpoint rectangular approximation method; the method of approximating a definite integral over an interval, using the function values at the right-hand endpoints of the subintervals determined by a partition

S

- **secant line to a curve –** a line which passes through at least two points on a curve



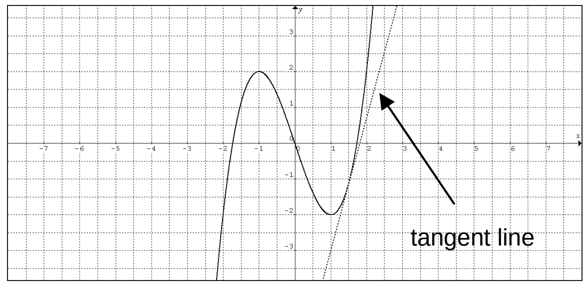
- **second derivative –** if *y* is a function of *x* and $y' = \frac{dy}{dx}$ is the first derivative of *y* with respect to *x*, then $y'' = \frac{dy'}{dx} = \frac{d^2y}{dx}$ is the second derivative of *y* with respect to *x*
- **Second Derivative Test (for Local Extrema)**
  - If $f'(c) = 0$ and $f''(c) < 0,$ then *f* has a local maximum at $x = c$.
  - If $f'(c) = 0$ and $f''(c) > 0,$ then *f* has a local minimum at $x = c$.
- **sensitivity of a function –** a measure of the rate of change of a function
- **sigma notation –** notation using the Greek letter capitol sigma, $\sum$, for writing lengthy sums in compact form
- **slope of a curve –** the slope of the curve $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$, provided *f* is differentiable at *a*
- **speed –** the magnitude of velocity, $v(t)$,

$$\text{speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

- **stationary point –** a point *c* in the domain of *f* at which $f'(c) = 0$
- **subinterval –** a part of the interval $[a, b]$, such that the left and right side of the subinterval are consecutive points of a partition

T

- **tangent line to a curve –** if a function $y = f(x)$ is differentiable at $x = a$, then a line is tangent to the graph of *f* at $(a, f(a))$ provided that the line passes through $(a, f(a))$ and the slope of the line is $f'(a)$



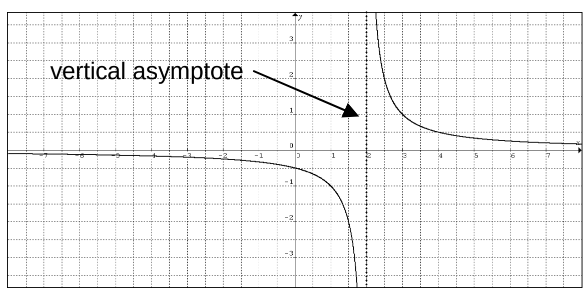
- **two-sided limit –** a limit at an interior point of a function's domain

U

- ***u*-substitution –** see **antidifferentiation by substitution**

V

- **variable of integration –** in $\int f(x)dx$ or $\int_a^b f(x)dx$, the variable *x*
- **velocity –** the rate of change of position with respect to time
- **vertical asymptote –** the line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$



SOLUTIONS TO POSSIBLE ASSESSMENT STRATEGIES

SECTION 2.1

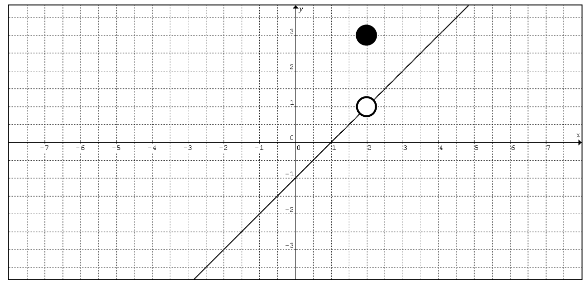
- a. 14.7 m/s
  b. 29.4 m/s
- a. 9
  b. 4
  c. $-\frac{1}{16}$
  d. –10
  e. $\frac{2}{3}$
- a. 3
  b. 1
  c. does not exist

SECTION 2.2

- horizontal asymptote: $y = 2$; vertical asymptotes: $x = \pm 1$
- a. 0
  b. $-\infty$
  c. $\frac{5}{2}$
- a. $y = 2x^2$
  b. $y = \frac{1}{2}x$
  c. $y = -1$
- $y = 3x^2$

SECTION 2.3

- a. $x = -2$; infinite discontinuity
  b. $x = 1$; jump discontinuity
- a. $y = \begin{cases} \frac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$
  b. $y = \begin{cases} \frac{\cos x - 1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
- Solutions may vary. One possible solution is

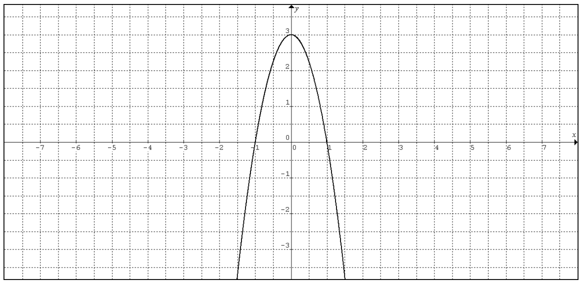


SECTION 2.4

- a. –11
  b. $\frac{\sqrt{2}}{2}$
  c. $\frac{1}{5}$
- a. slope: 2; tangent line: $y = 2x + 1$; normal line: $y = -\frac{1}{2}x + \frac{7}{2}$
  b. slope: 1; tangent line: $y = x$; normal line: $y = -x$
  c. slope: –2; tangent line: $y = -2x - 6$; normal line: $y = \frac{1}{2}x - 1$
- –4 m/s
- 10 cm^2/cm

SECTION 3.1

- a. –13
  b. $-\frac{1}{16}$
  c. –2
- a. $f'(x) = 2x - 6x^2$
  b. $f'(x) = \frac{-1}{2\sqrt{9 - x}}$
  c. $f'(x) = \frac{-7}{(3 + x)^2}$

- 

SECTION 3.2

- $x = 0$: discontinuity; $x = 2$: corner
- a. cusp
  b. corner
  c. discontinuity
  d. vertical tangent
- a. $x = 5$
  b. $x = 3$
  c. $x = -2$
  d. $x = 0$

SECTION 3.3

- a. $f'(x) = 3x^5 - 12x^3 + 1$
  b. $f'(x) = -6x^5 - 5x^4 - 4x^3 + 4x + 2$
  c. $f'(x) = 1 + x^{-2} + 6x^{-4}$
  d. $f'(x) = \dfrac{-x^2 - 2x - 3}{\left(x^2 + x - 2\right)^2}$
- a. $y = 12x - 3$
  b. $y = \dfrac{1}{2}x - \dfrac{1}{2}$
- a. $y' = 4x^3 - 9x^2 + 16$; $y'' = 12x^2 - 18x$; $y''' = 24x - 18$
  b. $y' = \dfrac{1}{x^2}$; $y'' = \dfrac{-2}{x^3}$; $y''' = \dfrac{6}{x^4}$
- $y = 12x - 15$; $y = 12x + 17$

SECTION 3.4

- a. $v(t) = 3t^2 - 12t + 9$
  b. -3 m/s; 9 m/s
  c. 1 s and 3 s
  d. positive direction: $0 < t < 1$ or $t > 3$; negative direction: $1 < t < 3$
  e. $a(t) = 6t - 12$; 12 m/s^2
  f. speeding up: $1 < t < 2$ or $t > 3$; slowing down: $0 < t < 1$ or $2 < t < 3$
- a. 4.9 m/s, –14.7 m/s
  b. 2.5 s
  c. 30.625 m
  d. 5 s
  e. –24.5 m/s
- a. –218.75 L/min; –187.5 L/min; –125 L/min; 0 L/min
  b. 0 min; 40 min

SECTION 3.5

- a. $y' = 6x + 2 \sin x$
  b. $y' = \dfrac{2 - \tan x + x \sec^2 x}{(2 - \tan x)^2}$
  c. $y' = -x^2 \sin x + 2x \cos x$
  d. $y' = \dfrac{\sin x + x \cos x + x^2 \cos x}{(1 + x)^2}$
- $y = 2x$

SECTION 4.1

- a. $y' = 10x\left(2x^2 + 3\right)\left(x^4 + 3x^2 - 2\right)^4$
  b. $y' = (2x - 3)^3\left(x^2 + x + 1\right)^4\left(28x^2 - 12x - 7\right)$
  c. $y' = \dfrac{-12x\left(x^2 + 1\right)^2}{\left(x^2 - 1\right)^4}$
  d. $y = \dfrac{\sin x - 2x \cos x - 2 \cos x}{\sin^3 x}$
- a. 20
  b. –1

SECTION 4.2

- a. $\dfrac{dy}{dx} = \dfrac{9x}{y}$
  b. $\dfrac{dy}{dx} = \dfrac{-2x - y}{x - 2y}$

c. $\frac{dy}{dx}=\frac{2x+y\ \sin\ x}{\cos\ x-2y}$

d. $\frac{dy}{dx}=\frac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$

- a. –1

  b. $-\frac{1}{4}$

- $\left(-\frac{3\sqrt{2}}{2},-\frac{3\sqrt{2}}{2}\right),\ \left(\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right)$

- tangent line: $y=-\frac{1}{5}x+\frac{7}{5}$; normal line: $y=5x-9$

- $\frac{d^2y}{dx^2}=\frac{36y^2-81x^2}{16y^3}$

- a. $f'(x)=\frac{3x}{\sqrt{3x^2-1}}$

  b. $f'(x)=-\frac{2}{3}\ \cos\ x\ (\sin\ x)^{-5/3}$

  c. $f'(x)=\frac{-3x-5}{2\sqrt{x}(3x-5)^2}$

SECTION 1.6

- a. $x\approx 1.1593$

  b. $x\approx -0.8481$

  c. $x\approx 1.1071$

- a. $\frac{\pi}{3}$

  b. $\frac{\pi}{4}$

  c. $-\frac{\pi}{3}$

  d. $\frac{1}{2}$

  e. $\frac{2\sqrt{21}}{21}$

- a. $\sqrt{1-x^2}$

  b. $\frac{\sqrt{1-x^2}}{x}$

SECTION 4.3

- a. $y'=\frac{2x}{1+x^4}$

  b. $y'=\frac{-\sin\ x}{1+\cos^2 x}$

  c. $y'=\sin^{-1}x$

  d. $y'=\frac{1}{\sqrt{-x^2-x}}$

- $y=-\frac{2\sqrt{3}}{3}x+\frac{\pi+\sqrt{3}}{3}$

SECTION 4.4

- a. $f'(x)=10e^{2x}$

  b. $f'(x)=xe^{x-2}(x+2)$

  c. $f'(x)=e^x(x^3+3x^2+2x+2)$

  d. $f'(x)=e^x(\cos\ x+\sin\ x)$

  e. $f'(x)=\frac{2}{2x+3}$

  f. $f'(x)=1-6x+\ln\ x$

  g. $f'(x)=7^{3x^2}\bullet 6x\ \ln\ 7$

  h. $f'(x)=\frac{3}{(3x-1)\ln\ 5}$

- $y=5x+1$

- a. $f'(x)=x^{\sin\ x}\left(\frac{\sin\ x}{x}+\cos\ x\ \ln\ x\right)$

  b. $f'(x)=\frac{(3x+1)^4(x-1)^6}{\sqrt{6x-5}}\left(\frac{12}{3x+1}+\frac{6}{x-1}-\frac{3}{6x-5}\right)$

SECTION 5.1

- a. local maximum: $(-2,2)$; absolute minimum: $(-1,1)$; absolute maximum: $(5,37)$

  b. absolute maximum: $(1,1)$

  c. absolute maximum: $(-3,18)$; local minimum: $(-1,-2)$; local maximum: $(1,2)$; absolute minimum: $(3,-18)$

  d. absolute minimum: $(-\pi,-1)$; absolute maximum: $(0,1)$; absolute minimum: $(\pi,-1)$

- a. absolute maximum: $(1,7)$

b. absolute minimum: $(0,-1)$; absolute maximum: $\left(2,\frac{1}{3}\right)$

c. local maximum: $(-2,3)$; local minimum: $(0,-1)$

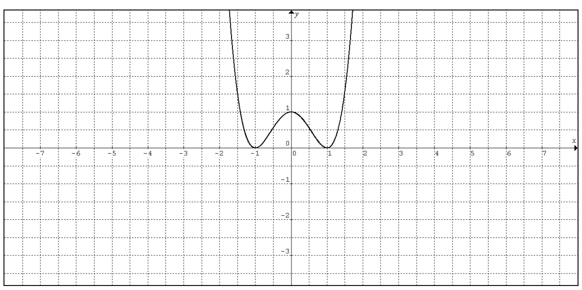
SECTION 5.2

- a. local maximum: $(-3,81)$; local minimum; $(2,-44)$; increasing: $x<-3$ and $x>2$; decreasing; $-3<x<2$

  b. absolute minimum: $(-1,2)$; local maximum; $(0,3)$; absolute minimum: $(1,2)$; increasing: $-1<x<0$ and $x>1$; decreasing; $x<-1$ and $0<x<1$

  c. absolute minimum: $\left(-1,-\frac{1}{2}\right)$; absolute maximum: $\left(1,\frac{1}{2}\right)$; increasing: $-1<x<1$; decreasing: $x<-1$ and $x>1$

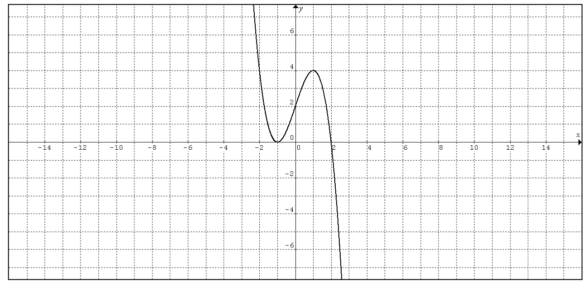
  d. absolute minimum: $(9,0)$; increasing: $x>9$
- Answers may vary. One possible solution is



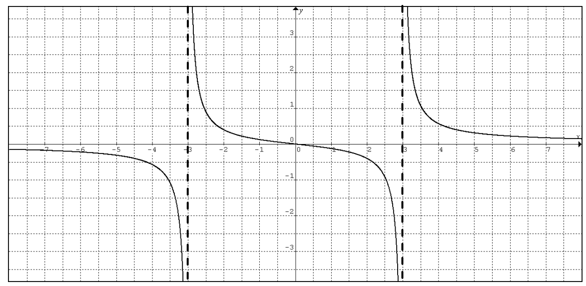
SECTION 5.3

- a. point of inflection: $(0,0)$; concave up: $x<0$; concave down: $x>0$

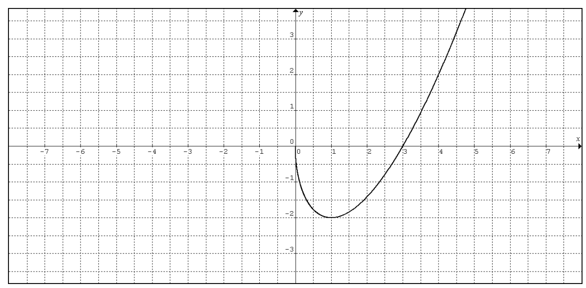
  b. point of inflection: $\left(\frac{1}{4},-\frac{5}{8}\right)$; concave up: $x>\frac{1}{4}$; concave down: $x<\frac{1}{4}$
- a.



b.



c.



SECTION 5.4

- 128
- 600 m by 1200 m
- $225
- $(2,2)$
- radius: $\sqrt[3]{\frac{500}{\pi}}$ m ≈ 5.4 m; height: $2\sqrt[3]{\frac{500}{\pi}}$ m ≈ 10.8 m

SECTION 5.5

- a. $L(x)=\frac{1}{4}x+\frac{7}{4}$

  b. $L(x)=-\frac{\sqrt{2}}{2}x+\frac{(\pi+4)\sqrt{2}}{8}$
- a. 10.9545

  b. 5.0133

- a. $dy = (2x - 4)dx$; 0.2

  b. $dy = \frac{dx}{(x+1)^2}$; 0.0125

- a. 2.0946

  b. 0.7391

SECTION 5.6

- $\frac{1}{25\pi}$ cm/s $\approx$ 0.0127 cm/s
- $\frac{3}{4}$ ft/s
- 78 mph
- $\frac{8}{9\pi}$ m/min $\approx$ 0.283 m/min
- 0.128 rad/s

SECTION 6.1

- a. 8.66

  b. 1.0971

  c. 2.0129

SECTION 6.2

- a. $\int_2^4 (x^2 + 3)\, dx$

  b. $\int_0^3 \frac{3}{x+1}\, dx$

- a. 35

  b. 16

- a. 18

  b. 2π

  c. $\frac{49}{2}$

SECTION 6.3.1

- a. –3

  b. 0

  c. 24

  d. 15

SECTION 6.3.2

- a. $f'(x) = \frac{1}{2}x^2 - 3x + C$

  b. $f'(x) = \frac{1}{2}x + \frac{1}{4}x^3 - \frac{1}{5}x^4 + C$

  c. $f'(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

  d. $f'(x) = 2 \sin x + C$

  e. $f'(x) = 5 \ln x + C$

- a. $f(x) = x^2 - 3x + 3$

  b. $f(x) = \frac{1}{6}x^3 - x^2 + 6x - 1$

- a. $\frac{1}{3}x^3 - x^{-1} + C$

  b. $\frac{1}{4}x^4 - 4\sqrt{x} + C$

  c. $\frac{2}{3}x^3 - \frac{5}{2}x^2 - 3x + C$

SECTION 6.4

- a. $\frac{dy}{dx} = \cos^2 x$

  b. $\frac{dy}{dx} = x^2 - 3x + 1$

  c. $\frac{dy}{dx} = -e^{3x}$

- a. $\frac{10}{3}$

  b. $\frac{9}{8}$

  c. $\frac{3}{4}$

  d. 63

  e. $\frac{52}{3}$

  f. $-\frac{27}{2}$

  g. $\frac{2+\sqrt{2}}{2}$

  h. $e - 1$

SECTION 7.2

- a. $\frac{2}{3}(1+x^2)^{3/2} + C$

b. $\frac{1}{4}\sin\left(x^4+2\right)+C$

- a. $\frac{26}{3}$

  b. $\frac{1}{14}$

SECTION 8.2

- a. $\frac{27}{2}$

  b. $\frac{46}{3}$

  c. $\frac{61}{4}$

  d. $\frac{25}{4}$

- a. $\frac{39}{2}$

  b. $\frac{3\pi^2}{8}-1$

  c. $\frac{9}{2}$

  d. $\frac{125}{6}$

APPENDIX

MATHEMATICS RESEARCH PROJECT

➢ Introduction

A research project can be a very important part of a mathematics education. Besides the greatly increased learning intensity that comes from personal involvement with a project, and the chance to show universities, colleges, and potential employers the ability to initiate and carry out a complex task, it gives the student an introduction to mathematics as it is – a living and developing intellectual discipline where progress is achieved by the interplay of individual creativity and collective knowledge. A major research project must successfully pass through several stages. Over the next few pages, these stages are highlighted, together with some ideas that students may find useful when working through such a project.

➢ Creating an Action Plan

As previously mentioned, a major research project must successfully pass though several stages. The following is an outline for an action plan, with a list of these stages, a suggested time, and space for students to include a probable time to complete each stage.

STAGE	SUGGESTED TIME	PROBABLE TIME
Select the topic to explore.	1 – 3 days	
Create the research question to be answered.	1 – 3 days	
Collect the data.	5 – 10 days	
Analyse the data.	5 – 10 days	
Create an outline for the presentation.	2 – 4 days	
Prepare a first draft.	3 – 10 days	
Revise, edit and proofread.	3 – 5 days	
Prepare and practise the presentation.	3 – 5 days	

Completing this action plan will help students organize their time and give them goals and deadlines that they can manage. The times that are suggested for each stage are only a guide. Students can adjust the time that they will spend on each stage to match the scope of their projects. For example, a project based on primary data (data that they collect) will usually require more time than a project based on secondary data (data that other people have collected and published). A student will also need to consider his or her personal situation – the issues that each student deals with that may interfere with the completion of his or her project. Examples of these issues may include:

- a part-time job;
- after-school sports and activities;
- regular homework;
- assignments for other courses;
- tests in other courses;
- time they spend with friends;
- family commitments;
- access to research sources and technology.

➢ Selecting the Research Topic

To decide what to research, a student can start by thinking about a subject and then consider specific topics. Some examples of subjects and topics may be:

SUBJECT	TOPIC
Entertainment	• effects of new electronic devices • file sharing
Health care	• doctor and/or nurse shortages • funding
Post-secondary education	• entry requirements • graduate success
History of Western and Northern Canada	• relations among First Nations • immigration

It is important to take the time to consider several topics carefully before selecting a topic for a research project. The following questions will help a student determine if a topic that is being considered is suitable.

- **Does the topic interest the student?**

  Students will be more successful if they choose a topic that interests them. They will be more motivated to do research, and they will be more attentive while doing the research. As well, they will care more about the conclusions they make.

- **Is the topic practical to research?**

  If a student decides to use first-hand data, can the data be generated in the time available, with the resources available? If a student decides to use second-hand data, are there multiple sources of data? Are the sources reliable, and can they be accessed in a timely manner?

- **Is there an important issue related to the topic?**

  A student should think about the issues related to the topic that he or she has chosen. If there are many viewpoints on an issue, they may be able to gather data that support some viewpoints but not others. The data they collect from all viewpoints should enable them to come to a reasoned conclusion.

- **Will the audience appreciate the presentation?**

  The topic should be interesting to the intended audience. Students should avoid topics that may offend some members of their audience.

➢ Creating the Research Question or Statement

A well-written research question or statement clarifies exactly what the project is designed to do. It should have the following characteristics:

- The research topic is easily identifiable.

- The purpose of the research is clear.
- The question or statement is focused. The people who are listening to or reading the question or statement will know what the student is going to be researching.

A good question or statement requires thought and planning. Below are three examples of initial questions or statements and how they were improved.

UNACCEPTABLE QUESTION OR STATEMENT	WHY?	ACCEPTABLE QUESTION OR STATEMENT
Is mathematics used in computer technology?	Too general	What role has mathematics played in the development of computer animation?
Water is a shared resource.	Too general	Homes, farms, ranches, and businesses east of the Rockies all use runoff water. When there is a shortage, that water must be shared.
Do driver's education programs help teenagers parallel park?	Too specific, unless the student is generating his or her own data	Do driver's education programs reduce the incidence of parking accidents?

The following checklist can be used to determine if the research question or statement is effective.

- Does the question or statement clearly identify the main objective of the research? After the question or statement is read to someone, can they tell what the student will be researching?
- Is the student confident that the question or statement will lead him or her to sufficient data to reach a conclusion?
- Is the question or statement interesting? Does it make the student want to learn more?
- Is the topic that the student chose purely factual, or is that student likely to encounter an issue, with different points of view?

➢ Carrying Out the Research

As students continue with their projects, they will need to conduct research and collect data. The strategies that follow will help them in their data collection.

There are two types of data that students will need to consider – primary and secondary. Primary data is data that the student collects himself or herself using surveys, interviews and direct observations. Secondary data is data that the student obtains through other sources, such as online publications, journals, magazines, and newspapers.

Both primary and secondary data have their advantages and disadvantages. Primary data provide specific information about the research question or statement, but may take time to collect and process. Secondary data is usually easier to obtain and can be analysed in less time. However, because the data was originally gathered for other purposes, a student may need to sift through it to find exactly what he or she is looking for.

The type of data chosen can depend on many factors, including the research question, the skills of the student, and available time and resources. Based on these and other factors, the student may chose to use primary data, secondary data, or both.

When collecting primary data, the student must ensure the following:

- For surveys, the sample size must be reasonably large and the random sampling technique must be well designed.
- For surveys and interviews, the questionnaires must be designed to avoid bias.
- For experiments and studies, the data must be free from measurement bias.
- The data must be compiled accurately.

When collecting secondary data, the student should explore a variety of resources, such as:

- textbooks, and other reference books;
- scientific and historical journals, and other expert publications;
- the Internet;
- library databases.

After collecting the secondary data, the student must ensure that the source of the data is reliable:

- If the data is from a report, determine what the author's credentials are, how recent the data is, and whether other researchers have cited the same data.
- Be aware that data collection is often funded by an organization with an interest in the outcome or with an agenda that it is trying to further. Knowing which organization has funded the data collection may help the student decide how reliable the data is, or what type of bias may have influenced the collection or presentation of the data.
- If the data is from the Internet, check it against the following criteria:
    - authority – the credentials of the author should be provided;
    - accuracy – the domain of the web address may help the student determine the accuracy;
    - currency – the information is probably being accurately managed if pages on a site are updated regularly and links are valid.

➢ Analysing the Data

Statistical tools can help a student analyse and interpret the data that is collected. Students need to think carefully about which statistical tools to use and when to use them, because other people will be scrutinizing the data. A summary of relevant tools follows.

Measures of central tendency will give information about which values are representative of the entire set of data. Selecting which measure of central tendency (mean, median, or mode) to use depends on the distribution of the data. As the researcher, the student must decide which measure most accurately describes the tendencies of the population. The following criteria should be considered when deciding upon which measure of central tendency best describes a set of data.

- Outliers affect the mean the most. If the data includes outliers, the student should use the median to avoid misrepresenting the data. If the student chooses to use the mean, the outliers should be removed before calculating the mean.

- If the distribution of the data is not symmetrical, but instead strongly skewed, the median may best represent the set of data.
- If the distribution of the data is roughly symmetrical, the mean and median will be close, so either may be appropriate to use.
- If the data is not numeric (for example, colour), or if the frequency of the data is more important than the values, use the mode.

Both the range and the standard deviation will give the student information about the distribution of data in a set. The range of a set of data changes considerably because of outliers. The disadvantage of using the range is that it does not show where most of the data in a set lies – it only shows the spread between the highest and the lowest values. The range is an informative tool that can be used to supplement other measures, such as standard deviation, but it is rarely used as the only measure of dispersion.

Standard deviation is the measure of dispersion that is most commonly used in statistical analysis when the mean is used to calculate central tendency. It measures the spread relative to the mean for most of the data in the set. Outliers can affect standard deviation significantly. Standard deviation is a very useful measure of dispersion for symmetrical distributions with no outliers. Standard deviation helps with comparing the spread of two sets of data that have approximately the same mean. For example, the set of data with the smaller standard deviation has a narrower spread of measurement around the mean, and therefore has comparatively fewer high or low scores, than a set of data with a higher standard deviation.

When working with several sets of data that approximate normal distributions, you can use *z*-scores to compare the data values. A *z*-score table enables a student to find the area under a normal distribution curve with a mean of zero and a standard deviation of one. To determine the *z*-score for any data value in a set that is normally distributed, the formula $z = \frac{x - \bar{x}}{s}$ can be used where x is any observed data value in the set, $\bar{x}$ is the mean of the set, and is s is the standard deviation of the set.

When analysing the results of a survey, a student may need to interpret and explain the significance of some additional statistics. Most surveys and polls draw their conclusions from a sample of a larger group. The margin of error and the confidence level indicate how well a sample represents a larger group. For example, a survey may have a margin of error of plus or minus 3% at a 95% level of confidence. This means that if the survey were conducted 100 times, the data would be within 3 percent points above or below the reported results in 95 of the 100 surveys.

The size of the sample that is used for a poll affects the margin of error. If a student is collecting data, he or she must consider the size of the sample that is needed for a desired margin of error.

➢ Identifying Controversial Issues

While working on a research project, a student may uncover some issues on which people disagree. To decide on how to present an issue fairly, he or she should consider some questions to ask as the research proceeds.

- **What is the issue about?**

  The student should identify which type of controversy has been uncovered. Almost all controversy revolves around one or more of the following:
  - Values – What should be? What is best?
  - Information – What is the truth? What is a reasonable interpretation?

    - Concepts – What does this mean? What are the implications?

- **What positions are being taken on the issue?**

  The student should determine what is being said and whether there is reasonable support for the claims being made. Some questions to ask are:
    - Would you like that done to you?
    - Is the claim based on a value that is generally shared?
    - Is there adequate information?
    - Are the claims in the information accurate?
    - Are those taking various positions on the issue all using the same term definitions?

- **What is being assumed?**

  Faulty assumptions reduce legitimacy. The student can ask:
    - What are the assumptions behind an argument?
    - Is the position based on prejudice or an attitude contrary to universally held human values, such as those set out in the United Nations Declaration of Human Rights?
    - Is the person who is presenting a position or an opinion an insider or an outsider?

- **What are the interests of those taking positions?**

  The student should try to determine the motivations of those taking positions on the issue. What are their reasons for taking their positions? The degree to which the parties involved are acting in self-interest could affect the legitimacy of their opinions.

➢ The Final Product and Presentation

The final presentation should be more than just a factual written report of the information gathered from the research. To make the most of the student's hard work, he or she should select a format for the final presentation that suits his or her strengths, as well as the topic.

To make the presentation interesting, a format should be chosen that suits the student's style. Some examples are:

- a report on an experiment or an investigation;
- a short story, musical performance, or play;
- a web page;
- a slide show, multimedia presentation, or video;
- a debate;
- an advertising campaign or pamphlet;
- a demonstration or the teaching of a lesson.

Sometimes, it is also effective to give the audience an executive summary of the presentation. This is a one-page summary of the presentation that includes the research question and the conclusions that were made.

Before giving the presentation, the student can use these questions to decide if the presentation will be effective.

- Did I define my topic well? What is the best way to define my topic?
- Is my presentation focused? Will my classmates find it focused?
- Did I organize my information effectively? Is it obvious that I am following a plan in my presentation?
- Am I satisfied with my presentation? What might make it more effective?
- What unanswered questions might my audience have?

➢ Peer Critiquing of Research Projects

After the student has completed his or her research for the question or statement being studied, and the report and presentation have been delivered, it is time to see and hear the research projects developed by other students. However, rather than being a passive observer, the student should have an important role – to provide feedback to his or her peers about their projects and presentations.

Critiquing a project does not involve commenting on what might have been or should have been. It involves reacting to what is seen and heard. In particular, the student should pay attention to:

- strengths and weaknesses of the presentation;
- problems or concerns with the presentation.

While observing each presentation, students should consider the content, the organization, and the delivery. They should take notes during the presentation, using the following rating scales as a guide. These rating scales can also be used to assess the presentation.

Content

Shows a clear sense of audience and purpose.	1	2	3	4	5
Demonstrates a thorough understanding of the topic.	1	2	3	4	5
Clearly and concisely explains ideas.	1	2	3	4	5
Applies knowledge and skills developed in this course.	1	2	3	4	5
Justifies conclusions with sound reasoning.	1	2	3	4	5
Uses vocabulary, symbols and diagrams correctly.	1	2	3	4	5

Organization

Presentation is clearly focused.	1	2	3	4	5
Engaging introduction includes the research question, clearly stated.	1	2	3	4	5
Key ideas and information are logically presented.	1	2	3	4	5
There are effective transitions between ideas and information.	1	2	3	4	5
Conclusion follows logically from the analysis and relates to the question.	1	2	3	4	5

Delivery

Speaking voice is clear, relaxed, and audible.	1	2	3	4	5
Pacing is appropriate and effective for the allotted time.	1	2	3	4	5
Technology is used effectively.	1	2	3	4	5
Visuals and handouts are easily understood.	1	2	3	4	5
Responses to audience's questions show a thorough understanding of the topic.	1	2	3	4	5

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