Prince Edward Island
CANADA

Education and Early Years

Prince Edward Island Mathematics Curriculum

Mathematics

MAT621B

CURRICULUM

Acknowledgments

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Background and Rationale

The development of an effective mathematics curriculum has encompassed a solid research base. Developers have examined the curriculum proposed throughout Canada and secured the latest research in the teaching of mathematics, and the result is a curriculum that should enable students to understand and use mathematics.

The Western and Northern Canadian Protocol (WNCP) *Common Curriculum Framework for Grades 10-12 Mathematics* (2008) has been adopted as the basis for a revised high school mathematics curriculum in Prince Edward Island. The *Common Curriculum Framework* was developed by the seven Canadian western and northern ministries of education (British Columbia, Alberta, Saskatchewan, Manitoba, Yukon Territory, Northwest Territories, and Nunavut) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and achievement indicators agreed upon by the seven jurisdictions. This document is based on both national and international research by the WNCP, and on the *Principles and Standards for School Mathematics* (2000), published by the National Council of Teachers of Mathematics (NCTM).

➢ Essential Graduation Learnings

Essential Graduation Learnings (EGLs) are statements describing the knowledge, skills, and attitudes expected of all students who graduate from high school. Achievement of the essential graduation learnings will prepare students to continue to learn throughout their lives. These learnings describe expectations not in terms of individual school subjects but in terms of knowledge, skills, and attitudes developed throughout the curriculum. They confirm that students need to make connections and develop abilities across subject boundaries if they are to be ready to meet the shifting and ongoing demands of life, work, and study today and in the future. Essential graduation learnings are cross curricular, and curriculum in all subject areas is focused to enable students to achieve these learnings. Essential graduation learnings serve as a framework for the curriculum development process.

Specifically, graduates from the public schools of Prince Edward Island will demonstrate knowledge, skills, and attitudes expressed as essential graduation learnings, and will be expected to

- respond with critical awareness to various forms of the arts, and be able to express themselves through the arts;
- assess social, cultural, economic, and environmental interdependence in a local and global context;
- use the listening, viewing, speaking, and writing modes of language(s), and mathematical and scientific concepts and symbols, to think, learn, and communicate effectively;
- continue to learn and to pursue an active, healthy lifestyle;
- use the strategies and processes needed to solve a wide variety of problems, including those requiring language, and mathematical and scientific concepts;
- use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems.

More specifically, curriculum outcome statements articulate what students are expected to know and be able to do in particular subject areas. Through the achievement of curriculum outcomes, students demonstrate the essential graduation learnings.

➢ Curriculum Focus

There is an emphasis in the Prince Edward Island mathematics curriculum on particular key concepts at each grade which will result in greater depth of understanding. There is also more emphasis on number sense and operations in the early grades to ensure students develop a solid foundation in numeracy. The intent of this document is to clearly communicate to all educational partners high expectations for students in mathematics education. Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge (NCTM, *Principles and Standards for School Mathematics*, 2000).

The main goals of mathematics education are to prepare students to

- use mathematics confidently to solve problems;
- communicate and reason mathematically;
- appreciate and value mathematics;
- make connections between mathematics and its applications;
- commit themselves to lifelong learning;
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will

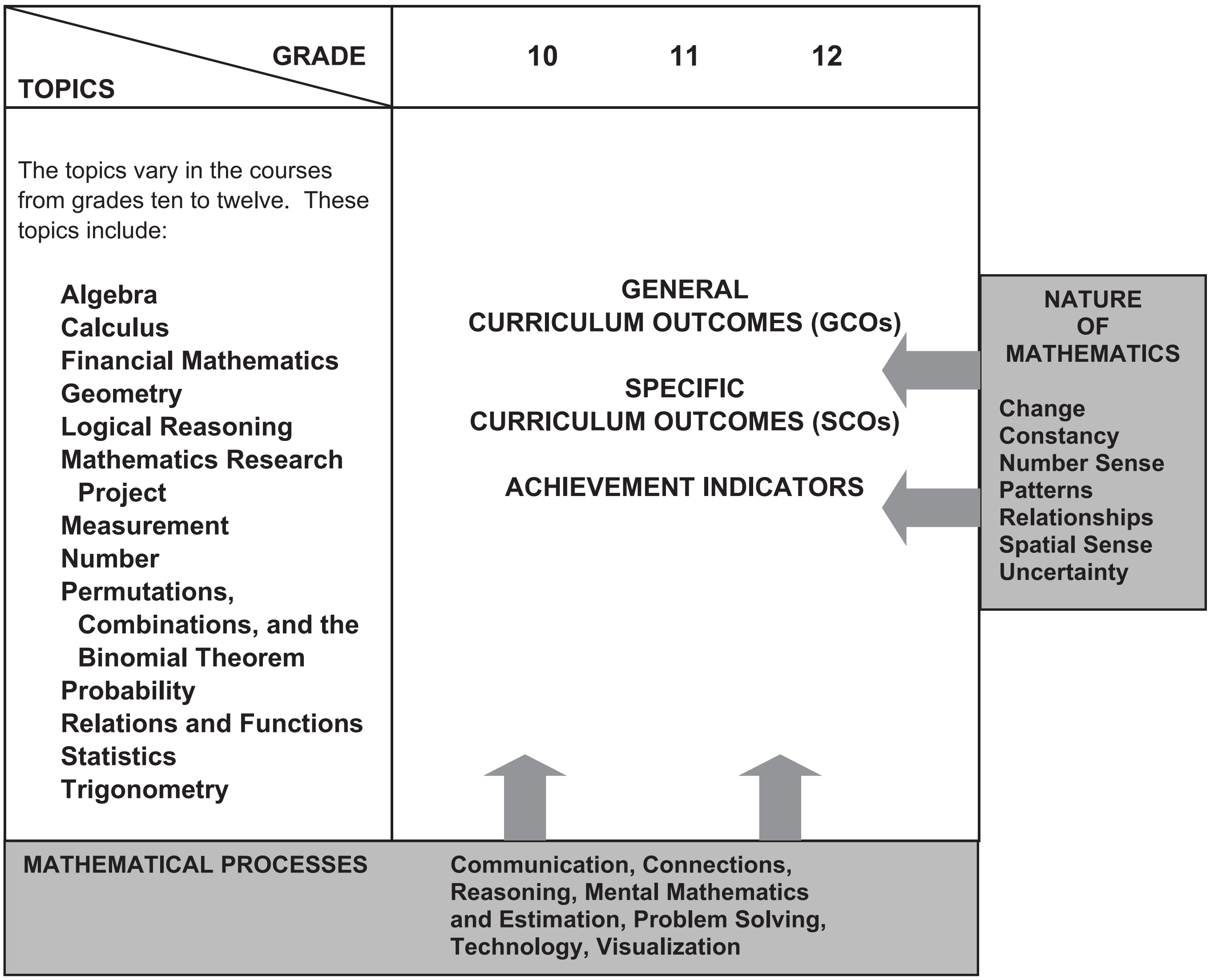
- gain understanding and appreciation of the contributions of mathematics as a science, philosophy, and art;
- exhibit a positive attitude toward mathematics;
- engage and persevere in mathematical tasks and projects;
- contribute to mathematical discussions;
- take risks in performing mathematical tasks;
- exhibit curiosity.

➢ Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the understanding of mathematical concepts by students and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in literacy, science, social studies, music, art, physical education, and other subject areas. Efforts should be made to make connections and use examples drawn from a variety of disciplines.

Conceptual Framework for 10-12 Mathematics

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



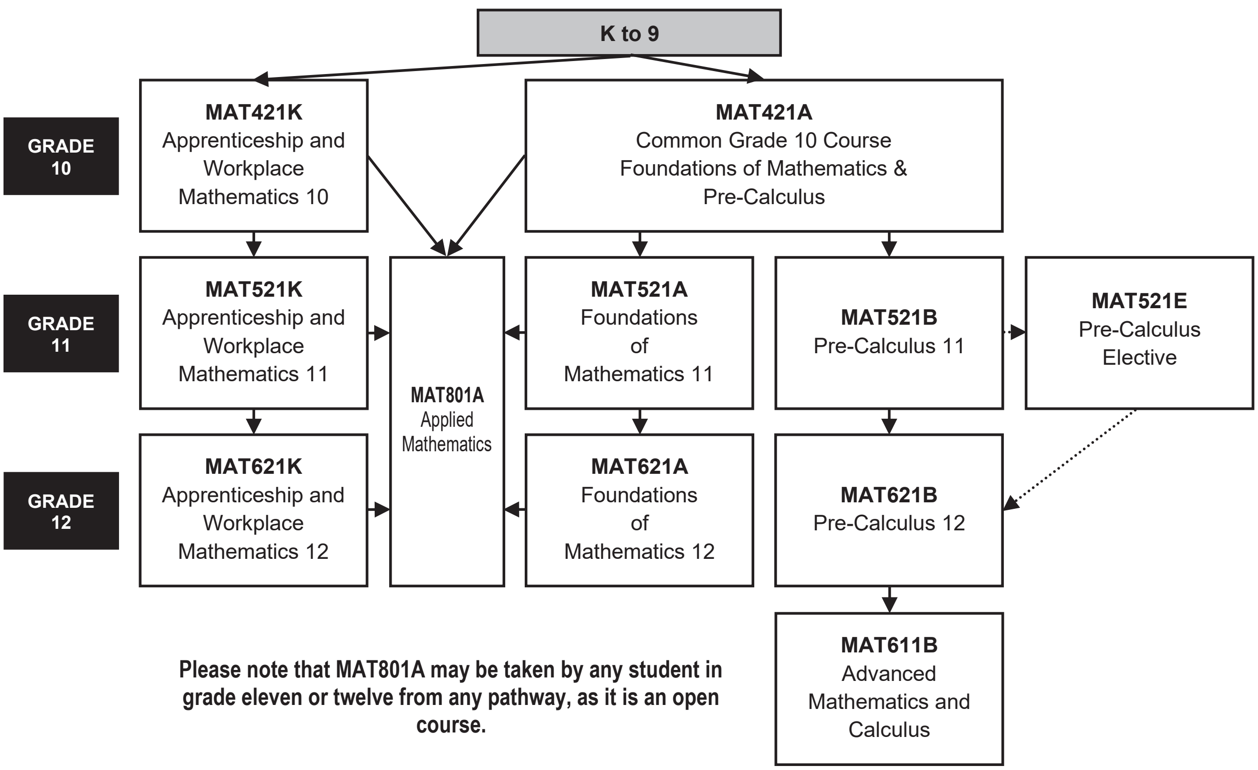
The mathematics curriculum describes the nature of mathematics, as well as the mathematical processes and the mathematical concepts to be addressed. This curriculum is arranged into a number of topics, as described above. These topics are not intended to be discrete units of instruction. The integration of outcomes across topics makes mathematical experiences meaningful. Students should make the connections among concepts both within and across topics. Consider the following when planning for instruction:

- Integration of the mathematical processes within each topic is expected.
- Decreasing emphasis on rote calculation, drill, and practice, and the size of numbers used in paper and pencil calculations makes more time available for concept development.

- Problem solving, reasoning, and connections are vital to increasing mathematical fluency, and must be integrated throughout the program.
- There is to be a balance among mental mathematics and estimation, paper and pencil exercises, and the use of technology, including calculators and computers. Concepts should be introduced using models and gradually developed from the concrete to the pictorial to the symbolic.

➢ Pathways and Topics

The Prince Edward Island 10-12 mathematics curriculum includes pathways with corresponding topics rather than strands, which are found in the Prince Edward Island K-9 mathematics curriculum. Three pathways are available: **Apprenticeship and Workplace Mathematics**, **Foundations of Mathematics**, and **Pre-Calculus**. A common grade ten course (Foundations of Mathematics and Pre-Calculus, Grade 10) is the starting point for the Foundations of Mathematics pathway and the Pre-Calculus pathway. Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings. These pathways are illustrated in the diagram below:



The goals of all three pathways are to provide the prerequisite knowledge, skills, understandings, and attitudes for specific post-secondary programs or direct entry into the work force. All three pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. Each pathway is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific post-secondary programs of study or for direct entry into the work force. When choosing a pathway, students should consider their interests, both current and future. Students, parents and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Apprenticeship and Workplace Mathematics

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into the majority of trades and for direct entry into the work force. Topics include algebra, geometry, measurement, number, statistics, and probability.

Foundations of Mathematics

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, measurement, algebra and number, logical reasoning, relations and functions, statistics, probability, and a mathematics research project.

Pre-Calculus

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Topics include algebra and number, measurement, relations and functions, trigonometry, combinatorics, and introductory calculus.

➢ Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics. The Prince Edward Island mathematics curriculum incorporates the following seven interrelated mathematical processes that are intended to permeate teaching and learning. These unifying concepts serve to link the content to methodology.

Students are expected to

- communicate in order to learn and express their understanding of mathematics; **[Communications: C]**
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines; **[Connections: CN]**
- demonstrate fluency with mental mathematics and estimation; **[Mental Mathematics and Estimation: ME]**
- develop and apply new mathematical knowledge through problem solving; **[Problem Solving: PS]**
- develop mathematical reasoning; **[Reasoning: R]**
- select and use technologies as tools for learning and solving problems; **[Technology: T]**
- develop visualization skills to assist in processing information, making connections, and solving problems. **[Visualization: V]**

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology. Communication can help students make connections among concrete, pictorial, symbolic, verbal, written, and mental representations of mathematical ideas.

Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

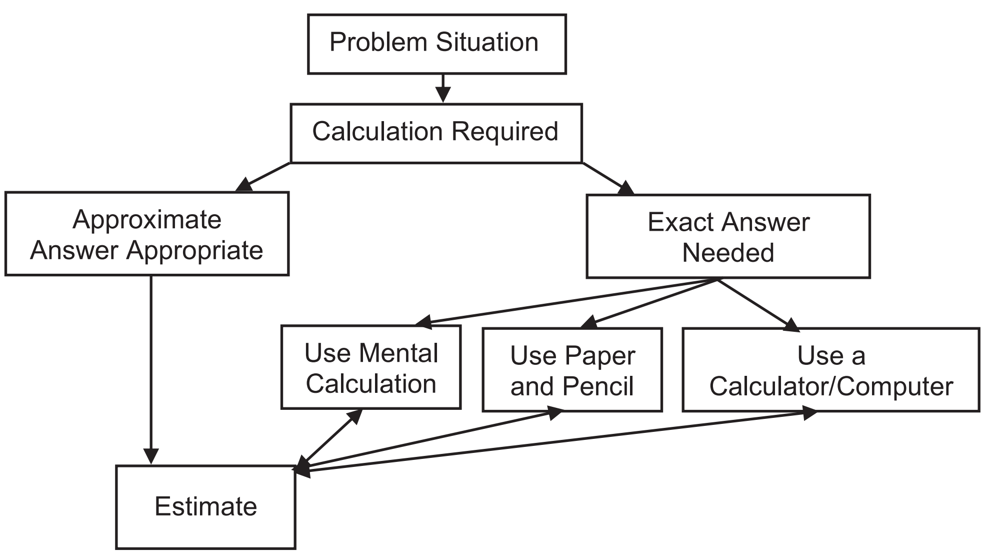
For instance, opportunities should be created frequently to link mathematics and career opportunities. Students need to become aware of the importance of mathematics and the need for mathematics in many career paths. This realization will help maximize the number of students who strive to develop and maintain the mathematical abilities required for success in further areas of study.

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It involves calculation without the use of external memory aids. Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy, and flexibility. Even more important than performing computational procedures or using calculators is the greater facility that students need - more than ever before - with estimation and mental mathematics (National Council of Teachers of Mathematics, May 2005). Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving” (Rubenstein, 2001). Mental mathematics “provides a cornerstone for all estimation processes offering a variety of alternate algorithms and non-standard techniques for finding answers” (Hope, 1988).

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know when to estimate, what strategy to use, and how to use it. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process described below:



(NCTM)

Problem Solving [PS]

Learning through problem solving should be the focus of mathematics at all grade levels. When students encounter new situations and respond to questions of the type, “How would you. . . ?" or “How could you. . . ?” the problem-solving approach is being modelled. Students develop their own problem-solving strategies by being open to listening, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement.

Problem solving is also a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for and engage in finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident and cognitive mathematical risk takers.

Over time, numerous problem-solving strategies should be modelled for students, and students should be encouraged to employ various strategies in many problem-solving situations. While choices with respect to the timing of the introduction of any given strategy will vary, the following strategies should all become familiar to students:

- using estimation
- guessing and checking
- looking for a pattern
- making an organized list or table
- using a model
- working backwards
- using a formula
- using a graph, diagram, or flow chart
- solving a simpler problem
- using algebra.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. High-order questions challenge students to think and develop a sense of wonder about mathematics. Mathematical experiences in and out of the classroom provide opportunities for inductive and deductive reasoning. Inductive reasoning occurs when students explore and record results, analyse observations, make generalizations from patterns, and test these generalizations. Deductive reasoning occurs when students reach new conclusions based upon what is already known or assumed to be true.

Technology [T]

Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Calculators and computers can be used to

- explore and demonstrate mathematical relationships and patterns;
- organize and display data;
- extrapolate and interpolate;
- assist with calculation procedures as part of solving problems;
- decrease the time spent on computations when other mathematical learning is the focus;
- reinforce the learning of basic facts and test properties;
- develop personal procedures for mathematical operations;
- create geometric displays;
- simulate situations;

- develop number sense.

Technology contributes to a learning environment in which the growing curiosity of students can lead to rich mathematical discoveries at all grade levels. While technology can be used in K-3 to enrich learning, it is expected that students will meet all outcomes without the use of technology.

Visualization [V]

Visualization involves thinking in pictures and images, and the ability to perceive, transform, and recreate different aspects of the visual-spatial world. The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret, and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure and when to estimate, and knowledge of several estimation strategies (Shaw & Cliatt, 1989).

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations.

➢ The Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics which are woven throughout this document. These components include change, constancy, number sense, patterns, relationships, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4;
- an arithmetic sequence, with first term 4 and a common difference of 2; or
- a linear function with a discrete domain.

Constancy

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state, and symmetry (AAAS–Benchmarks, 1993, p. 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180^0.

- The theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations, or the angle sums of polygons.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy (*The Primary Program*, B.C., 2000, p. 146). A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Number sense develops when students connect numbers to real-life experiences, and use benchmarks and referents. This results in students who are computationally fluent, and flexible and intuitive with numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all topics in mathematics and it is important that connections are made among topics. Working with patterns enables students to make connections within and beyond mathematics. These skills contribute to students' interaction with and understanding of their environment. Patterns may be represented in concrete, visual, or symbolic form. Students should develop fluency in moving from one representation to another. Students must learn to recognize, extend, create, and use mathematical patterns. Patterns allow students to make predictions and justify their reasoning when solving routine and non-routine problems. Learning to work with patterns in the early grades helps develop students' algebraic thinking that is foundational for working with more abstract mathematics in higher grades.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, and concepts. The search for possible relationships involves the collecting and analysing of data, and describing relationships visually, symbolically, orally, or in written form.

Spatial Sense

Spatial sense involves visualization, mental imagery, and spatial reasoning. These skills are central to the understanding of mathematics. Spatial sense enables students to interpret representations of 2-D shapes and 3-D objects, and identify relationships to mathematical topics. Spatial sense is developed through a variety of experiences and interactions within the environment. The development of spatial sense enables students to solve problems involving 2-D shapes and 3-D objects.

Spatial sense offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations. Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to use dimensions and make predictions about the results of changing dimensions.

- Knowing the dimensions of an object enables students to communicate about the object and create representations.
- The volume of a rectangular solid can be calculated from given dimensions.
- Doubling the length of the side of a square increases the area by a factor of four.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately.

Contexts for Learning and Teaching

The Prince Edward Island mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice:

- Mathematics learning is an active and constructive process.
- Learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates.
- Learning is most likely to occur in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and that nurtures positive attitudes and sustained effort.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Young children develop a variety of mathematical ideas before they enter school. They make sense of their environment through observations and interactions at home and in the community. Their mathematics learning is embedded in everyday activities, such as playing, reading, storytelling, and helping around the home. Such activities can contribute to the development of number and spatial sense in children. Initial problem solving and reasoning skills are fostered when children are engaged in activities such as comparing quantities, searching for patterns, sorting objects, ordering objects, creating designs, building with blocks, and talking about these activities. Positive early experiences in mathematics are as critical to child development as are early literacy experiences.

Students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. The use of models and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students, and enhance the formation of sound, transferable, mathematical concepts. At all levels, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful discussions can provide essential links among concrete, pictorial, and symbolic representations of mathematics.

The learning environment should value and respect the experiences and ways of thinking of all students, so that learners are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. Learners must be encouraged that it is acceptable to solve problems in different ways and realize that solutions may vary.

➢ Homework

Homework is an essential component of the mathematics program, as it extends the opportunity for students to think mathematically and to reflect on ideas explored during class time. The provision of this additional time for reflection and practice plays a valuable role in helping students to consolidate their learning.

Traditionally, homework has meant completing ten to twenty drill and practice questions relating to the procedure taught in a given day. With the increased emphasis on problem solving, conceptual understanding, and mathematical reasoning, however, it is important that homework assignments change accordingly. More assignments involving problem solving, mathematical investigations, written explanations and reflections, and data collection should replace some of the basic practice exercises given in isolation. In fact, a good problem can sometimes accomplish more than many drill-oriented exercises on a topic.

As is the case in designing all types of homework, the needs of the students and the purpose of the assignment will dictate the nature of the questions included. Homework need not be limited to reinforcing learning; it provides an excellent opportunity to revisit topics explored previously and to introduce new topics before teaching them in the classroom. Homework provides an effective way to communicate with parents and provides parents an opportunity to be actively involved in their child's learning. By ensuring that assignments model classroom instruction and sometimes require parental input, a teacher can give a parent clearer understanding of the mathematics curriculum and of the child's progress in relationship to it. As Van de Walle (1994, p. 454) suggests, homework can serve as a parent's window to the classroom.

➢ Diversity in Student Needs

Every class has students at many different cognitive levels. Rather than choosing a certain level at which to teach, a teacher is responsible for tailoring instruction to reach as many of these students as possible. In general, this may be accomplished by assigning different tasks to different students or assigning the same open-ended task to most students. Sometimes it is appropriate for a teacher to group students by interest or ability, assigning them different tasks in order to best meet their needs. These groupings may last anywhere from minutes to semesters, but should be designed to help all students (whether strong, weak or average) to reach their highest potential. There are other times when an appropriately open-ended task can be valuable to a broad spectrum of students. For example, asking students to make up an equation for which the answer is 5 allows some students to make up very simple equations while others can design more complex ones. The different equations constructed can become the basis for a very rich lesson from which all students come away with a better understanding of what the solution to an equation really means.

➢ Gender and Cultural Equity

The mathematics curriculum and mathematics instruction must be designed to equally empower both male and female students, as well as members of all cultural backgrounds. Ultimately, this should mean not only that enrolments of students of both genders and various cultural backgrounds in public school mathematics courses should reflect numbers in society, but also that representative numbers of both genders and the various cultural backgrounds should move on to successful post-secondary studies and careers in mathematics and mathematics-related areas.

➢ Mathematics for EAL Learners

The Prince Edward Island mathematics curriculum is committed to the principle that learners of English as an additional language (EAL) should be full participants in all aspects of mathematics education. English deficiencies and cultural differences must not be barriers to full participation. All students should study a comprehensive mathematics curriculum with high-quality instruction and co-ordinated assessment.

The *Principles and Standards for School Mathematics* (NCTM, 2000) emphasizes communication "as an essential part of mathematics and mathematics education" (p.60). The *Standards* elaborate that all students, and EAL learners in particular, need to have opportunities and be given encouragement and support for speaking, writing, reading, and listening in mathematics classes. Such efforts have the potential to help EAL learners overcome barriers and will facilitate “communicating to learn mathematics and learning to communicate mathematically” (NCTM, p.60).

To this end,

- schools should provide EAL learners with support in their dominant language and English language while learning mathematics;
- teachers, counsellors, and other professionals should consider the English-language proficiency level of EAL learners as well as their prior course work in mathematics;
- the mathematics proficiency level of EAL learners should be solely based on their prior academic record and not on other factors;
- mathematics teaching, curriculum, and assessment strategies should be based on best practices and build on the prior knowledge and experiences of students and on their cultural heritage;
- the importance of mathematics and the nature of the mathematics program should be communicated with appropriate language support to both students and parents;
- to verify that barriers have been removed, educators should monitor enrolment and achievement data to determine whether EAL learners have gained access to, and are succeeding in, mathematics courses.

➢ Education for Sustainable Development

Education for sustainable development (ESD) involves incorporating the key themes of sustainable development - such as poverty alleviation, human rights, health, environmental protection, and climate change - into the education system. ESD is a complex and evolving concept and requires learning about these key themes from a social, cultural, environmental, and economic perspective, and exploring how those factors are interrelated and interdependent.

With this in mind, it is important that all teachers, including mathematics teachers, attempt to incorporate these key themes in their subject areas. One tool that can be used is the searchable on-line database *Resources for Rethinking*, found at **http://r4r.ca/en**. It provides teachers with access to materials that integrate ecological, social, and economic spheres through active, relevant, interdisciplinary learning.

➢ Inquiry-Based Learning and Project Based Learning

Inquiry-based learning (IBL) allows students to explore, investigate, and construct new meaning from prior knowledge and from new information that is retrieved from other sources. It is not linear in nature, but promotes a continual looping back and forth throughout the process as students gather and process new information, redirect their inquiries, and continue through the process. Mathematical inquiry will require students to practise and refine their critical and creative-thinking skills. The terms *inquiry* and *research* are often used interchangeably within an educational context. While research often becomes the end-result of an inquiry process, it is the process itself that should be emphasized within an educational context. More information regarding the development of a mathematics research project is included in the appendix at the end of this document.

Assessment and Evaluation

Assessment and evaluation are essential components of teaching and learning in mathematics. The basic principles of assessment and evaluation are as follows:

- Effective assessment and evaluation are essential to improving student learning.
- Effective assessment and evaluation are aligned with the curriculum outcomes.
- A variety of tasks in an appropriate balance gives students multiple opportunities to demonstrate their knowledge and skills.
- Effective evaluation requires multiple sources of assessment information to inform judgments and decisions about the quality of student learning.
- Meaningful assessment data can demonstrate student understanding of mathematical ideas, student proficiency in mathematical procedures, and student beliefs and attitudes about mathematics.

Without effective assessment and evaluation it is impossible to know whether students have learned, teaching has been effective, or how best to address student learning needs. The quality of assessment and evaluation in the educational process has a profound and well-established link to student performance. Research consistently shows that regular monitoring and feedback are essential to improving student learning. What is assessed and evaluated, how it is assessed and evaluated, and how results are communicated send clear messages to students and others.

➢ Assessment

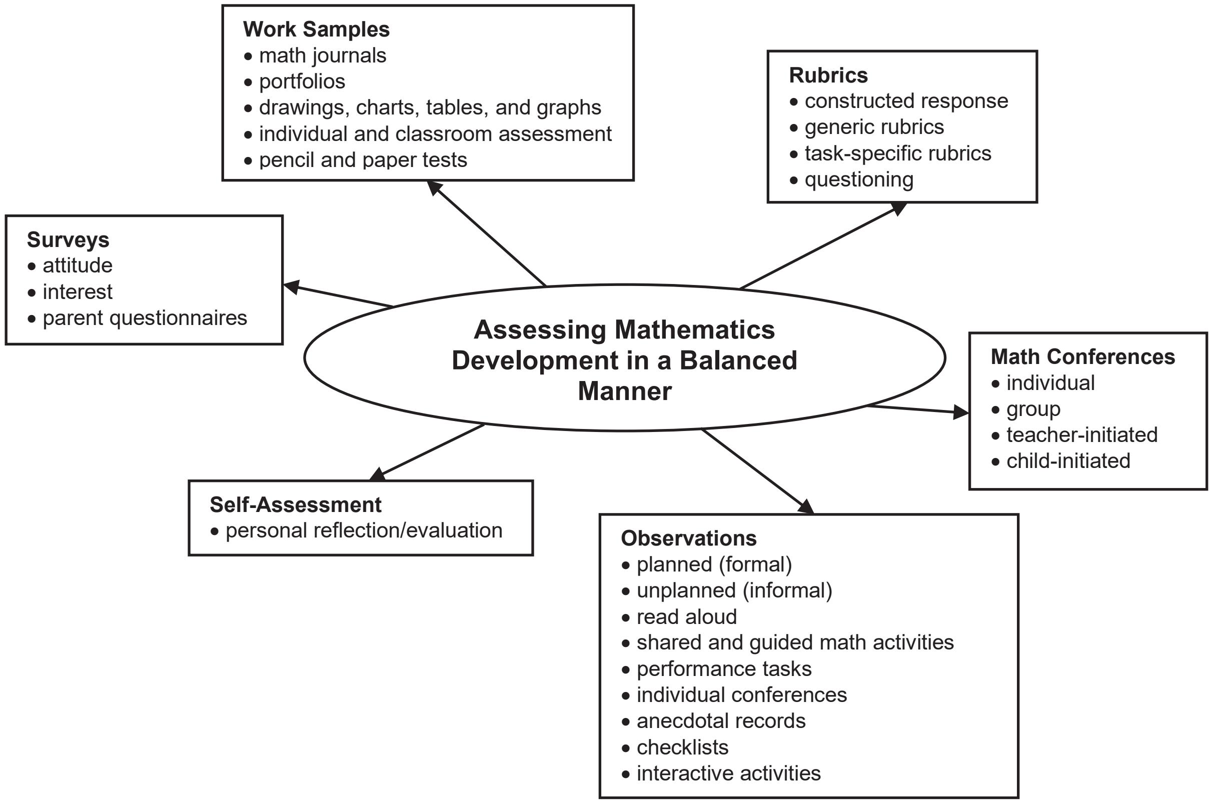
Assessment is the systematic process of gathering information on student learning. To determine how well students are learning, assessment strategies have to be designed to systematically gather information on the achievement of the curriculum outcomes. Teacher-developed assessments have a wide variety of uses, such as

- providing feedback to improve student learning;
- determining if curriculum outcomes have been achieved;
- certifying that students have achieved certain levels of performance;
- setting goals for future student learning;
- communicating with parents about their children's learning;
- providing information to teachers on the effectiveness of their teaching, the program, and the learning environment;
- meeting the needs of guidance and administration.

A broad assessment plan for mathematics ensures a balanced approach to summarizing and reporting. It should consider evidence from a variety of sources, including

- formal and informal observations
- work samples
- anecdotal records
- conferences
- teacher-made and other tests
- portfolios
- learning journals
- questioning
- performance assessment
- peer- and self-assessment.

This balanced approach for assessing mathematics development is illustrated in the diagram on the next page.



There are three interrelated purposes for classroom assessment: assessment *as* learning, assessment *for* learning, and assessment *of* learning. Characteristics of each type of assessment are highlighted below.

Assessment *as* learning is used

- to engage students in their own learning and self-assessment;
- to help students understand what is important in the mathematical concepts and particular tasks they encounter;
- to develop effective habits of metacognition and self-coaching;
- to help students understand themselves as learners - *how* they learn as well as *what* they learn - and to provide strategies for reflecting on and adjusting their learning.

Assessment *for* learning is used

- to gather and use ongoing information in relation to curriculum outcomes in order to adjust instruction and determine next steps for individual learners and groups;
- to identify students who are at risk, and to develop insight into particular needs in order to differentiate learning and provide the scaffolding needed;
- to provide feedback to students about how they are doing and how they might improve;
- to provide feedback to other professionals and to parents about how to support students' learning.

Assessment *of* learning is used

- to determine the level of proficiency that a student has demonstrated in terms of the designated learning outcomes for a unit or group of units;
- to facilitate reporting;

- to provide the basis for sound decision-making about next steps in a student's learning.

➢ Evaluation

Evaluation is the process of analysing, reflecting upon, and summarizing assessment information, and making judgments or decisions based upon the information gathered. Evaluation involves teachers and others in analysing and reflecting upon information about student learning gathered in a variety of ways.

This process requires

- developing clear criteria and guidelines for assigning marks or grades to student work;
- synthesizing information from multiple sources;
- weighing and balancing all available information;
- using a high level of professional judgment in making decisions based upon that information.

➢ Reporting

Reporting on student learning should focus on the extent to which students have achieved the curriculum outcomes. Reporting involves communicating the summary and interpretation of information about student learning to various audiences who require it. Teachers have a special responsibility to explain accurately what progress students have made in their learning and to respond to parent and student inquiries about learning. Narrative reports on progress and achievement can provide information on student learning which letter or number grades alone cannot. Such reports might, for example, suggest ways in which students can improve their learning and identify ways in which teachers and parents can best provide support. Effective communication with parents regarding their children's progress is essential in fostering successful home-school partnerships. The report card is one means of reporting individual student progress. Other means include the use of conferences, notes, phone calls, and electronic methods.

➢ Guiding Principles

In order to provide accurate, useful information about the achievement and instructional needs of students, certain guiding principles for the development, administration, and use of assessments must be followed. The document *Principles for Fair Student Assessment Practices for Education in Canada* (1993) articulates five fundamental assessment principles, as follows:

- Assessment methods should be appropriate for and compatible with the purpose and context of the assessment.
- Students should be provided with sufficient opportunity to demonstrate the knowledge, skills, attitudes, or behaviours being assessed.
- Procedures for judging or scoring student performance should be appropriate for the assessment method used and be consistently applied and monitored.
- Procedures for summarizing and interpreting assessment results should yield accurate and informative representations of a student's performance in relation to the curriculum outcomes for the reporting period.
- Assessment reports should be clear, accurate, and of practical value to the audience for whom they are intended.

These principles highlight the need for assessment which ensures that

- the best interests of the student are paramount;

- assessment informs teaching and promotes learning;
- assessment is an integral and ongoing part of the learning process and is clearly related to the curriculum outcomes;
- assessment is fair and equitable to all students and involves multiple sources of information.

While assessments may be used for different purposes and audiences, all assessments must give each student optimal opportunity to demonstrate what he or she knows and can do.

Structure and Design of the Curriculum Guide

The learning outcomes in the Prince Edward Island high school mathematics curriculum are organized into a number of topics across the grades from ten to twelve. Each topic has associated with it a general curriculum outcome (GCO). They are overarching statements about what students are expected to learn in each topic from grades ten to twelve.

Topic	General Curriculum Outcome (GCO)
Algebra (A)	Develop algebraic reasoning.
Algebra and Number (AN)	Develop algebraic reasoning and number sense.
Calculus (C)	Develop introductory calculus reasoning.
Financial Mathematics (FM)	Develop number sense in financial applications.
Geometry (G)	Develop spatial sense.
Logical Reasoning (LR)	Develop logical reasoning.
Mathematics Research Project (MRP)	Develop an appreciation of the role of mathematics in society.
Measurement (M)	Develop spatial sense and proportional reasoning. (***Foundations of Mathematics and Pre-Calculus***)
	Develop spatial sense through direct and indirect measurement. (***Apprenticeship and Workplace Mathematics***)
Number (N)	Develop number sense and critical thinking skills.
Permutations, Combinations and Binomial Theorem (PC)	Develop algebraic and numeric reasoning that involves combinatorics.
Probability (P)	Develop critical thinking skills related to uncertainty.
Relations and Functions (RF)	Develop algebraic and graphical reasoning through the study of relations.
Statistics (S)	Develop statistical reasoning.
Trigonometry (T)	Develop trigonometric reasoning.

Each general curriculum outcome is then subdivided into a number of specific curriculum outcomes (SCOs). Specific curriculum outcomes are statements that identify the specific skills, understandings, and knowledge students are required to attain by the end of a given grade.

Finally, each specific curriculum outcome has a list of achievement indicators that are used to determine whether students have met the corresponding specific curriculum outcome.

In this curriculum guide, each specific curriculum outcome (SCO) is presented in a two-page format, and includes the following information:

- its corresponding topic and general curriculum outcome;
- the scope and sequence of the specific curriculum outcome(s) from grades eleven to twelve which correspond to this SCO;
- the specific curriculum outcome, with a list of achievement indicators;

- a list of the sections in *Pre-Calculus 12* which address the SCO, with specific achievement indicators highlighted in brackets;
- an elaboration for the SCO.

TRIGONOMETRY

SPECIFIC CURRICULUM OUTCOMES

T1 – Demonstrate an understanding of angles in standard position, expressed in degrees and radians.

T2 – Develop and apply the equation of the unit circle.

T3 – Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

T4 – Graph and analyse the trigonometric functions sine, cosine and tangent to solve problems.

T5 – Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

T6 – Prove trigonometric identities, using:
- **reciprocal identities;**
- **quotient identities;**
- **Pythagorean identities;**
- **sum or difference identities (restricted to sine, cosine and tangent);**
- **double-angle identities (restricted to sine, cosine and tangent).**

Grade 12 – Topic: Trigonometry (T)

GCO: Develop trigonometric reasoning.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
T1 Demonstrate an understanding of angles in standard position (0^0 to 360^0).	**T1** Demonstrate an understanding of angles in standard position, expressed in degrees and radians.

SCO: **T1 – Demonstrate an understanding of angles in standard position, expressed in degrees and radians. [CN, ME, R, V]**

Students who have achieved this outcome should be able to:

A. Sketch, in standard position, an angle (positive or negative) when the measure is given in degrees.

B. Describe the relationship among different systems of angle measurement, with emphasis on radians and degrees.

C. Sketch, in standard position, an angle with a measure of 1 radian.

D. Sketch, in standard position, an angle with a measure expressed in the form $k\pi$ radians, where $k \in Q$.

E. Express the measure of an angle in radians (exact value or decimal approximation), given its measure in degrees.

F. Express the measure of an angle in degrees, given its measure in radians (exact value or decimal approximation).

G. Determine the measures, in degrees or radians, of all angles in a given domain that are coterminal with a given angle in standard position.

H. Determine the general form of the measures, in degrees or radians, of all angles that are coterminal with a given angle in standard position.

I. Explain the relationship between the radian measure of an angle in standard position and the length of the arc cut on a circle of radius *r*, and solve problems based upon that relationship.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

4.1 (A B C D E F G H I)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: T1 – Demonstrate an understanding of angles in standard position, expressed in degrees and radians. [CN, ME, R, V]

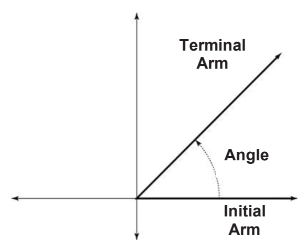
Elaboration

Angles can be measured in either degrees or radians. An angle measured in one unit can be converted to the other unit using the relationship $1 \text{ full rotation} = 360^0 = 2\pi$ radians. This relationship can be also written as follows:

$$1^0 = \frac{\pi}{180} \text{ radians}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^0$$

An angle in standard position has its vertex at the origin and its initial arm along the positive *x*-axis.

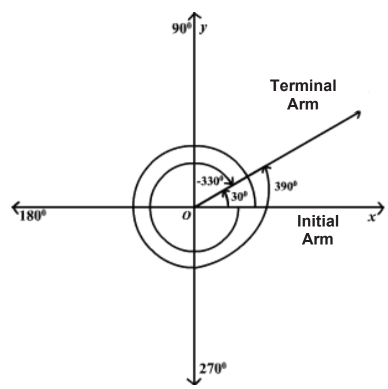


Angles that are coterminal have the same initial arm and the same terminal arm. An angle θ has an infinite number of angles that are coterminal to it, expressed by

$$\theta \pm (360n)^0 \text{ (in degrees), where } n \text{ is an integer}$$

or

$$\theta \pm 2\pi n \text{ (in radians), where } n \text{ is integer}$$



The arc length, *a*, of a circle can be found using the formula

$$a = \theta r$$

where θ is the central angle, and *r* is the length of the radius. In order to use this formula, *a* and *r* must be in the same units, and θ must be measured in radians.

Grade 12 – Topic: Trigonometry (T)

GCO: Develop trigonometric reasoning.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
T1 Demonstrate an understanding of angles in standard position (0^0 to 360^0).	**T2** Develop and apply the equation of the unit circle.

SCO: **T2 – Develop and apply the equation of the unit circle.** **[CN, R, V]**

Students who have achieved this outcome should be able to:

A. Derive the equation of the unit circle from the Pythagorean theorem.

B. Describe the six trigonometric ratios, using a point $P(x, y)$ that is the intersection of the terminal arm of an angle and the unit circle.

C. Generalize the equation of a circle with centre $(0,0)$ and radius r.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

4.2 (A C)

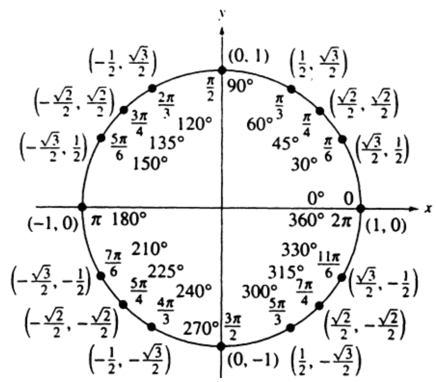
4.3 (B)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: T2 – Develop and apply the equation of the unit circle. [CN, R, V]

Elaboration

The equation of the unit circle is $x^2 + y^2 = 1$. It can be used to determine whether a point is on a unit circle or to determine the value of one coordinate, given the other coordinate. The equation for a circle with centre at $(0,0)$ and radius *r* is $x^2 + y^2 = r^2$. On the unit circle, the measure, in radians, of the central angle and the arc subtended by that central angle are numerically equivalent. Some of the points on the unit circle correspond to the exact values which correspond to the special angles in standard position learned previously.



Patterns can also be used to determine coordinates of points on the unit circle. For example, the numerical value of the coordinates of points on the unit circle change to their opposite sign every half-rotation. For example, if $P(\theta) = (a, b)$ is in Quadrant I, then both *a* and *b* are positive. Since $P(\theta + \pi)$ is in Quadrant III, its coordinates will be both negative, specifically, $P(\theta + \pi) = (-a, -b)$.

Each primary trigonometric ratio has a reciprocal trigonometric ratio, defined as follows:

$$\sec\ \theta = \frac{1}{\cos\ \theta} \qquad \csc\ \theta = \frac{1}{\sin\ \theta} \qquad \cot\ \theta = \frac{1}{\tan\ \theta}$$

Points that are on the intersection of the terminal arm of angle θ in standard position and the unit circle can be defined using trigonometric ratios, that is $P(\theta) = (x, y) = (\cos\ \theta, \sin\ \theta)$. As a result, we can determine the trigonometric ratios for any angle in standard position using the coordinates of the point where the terminal arm intersects the unit circle, (x, y):

$$\cos\ \theta = \frac{x}{1} = x \qquad \sin\ \theta = \frac{y}{1} = y \qquad \tan\ \theta = \frac{y}{x}$$

$$\sec\ \theta = \frac{1}{x} \qquad \csc\ \theta = \frac{1}{y} \qquad \cot\ \theta = \frac{x}{y}$$

Grade 12 – Topic: Trigonometry (T)

GCO: Develop trigonometric reasoning.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
T2 Solve problems, using the three primary trigonometric ratios for angles from 0^0 to 360^0 in standard position.	**T3** Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

SCO: **T3 – Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. [ME, PS, R, T, V]**

Students who have achieved this outcome should be able to:

A. Determine, with technology, the approximate value of a trigonometric ratio for any angle with a measure expressed in either degrees or radians.

B. Determine, using a unit circle or reference triangle, the exact value of a trigonometric ratio for angles expressed in degrees that are multiples of 0^0, 30^0, 45^0, 60^0 or 90^0, or for angles expressed in radians that are multiples of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ or $\frac{\pi}{2}$, and explain the strategy.

C. Determine, with or without technology, the measures, in degrees or radians, of the angles in a specified domain, given the value of a trigonometric ratio.

D. Explain how to determine the exact values of the six trigonometric ratios, given the coordinates of a point on the terminal arm of an angle in standard position.

E. Determine the measures of the angles in a specified domain in degrees or radians, given a point on the terminal arm of an angle in standard position.

F. Determine the exact values of the other trigonometric ratios, given the value of one trigonometric ratio in a specified domain.

G. Sketch a diagram to represent a problem that involves trigonometric ratios.

H. Solve a problem, using trigonometric ratios.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

4.3 (A B C D E F G H)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

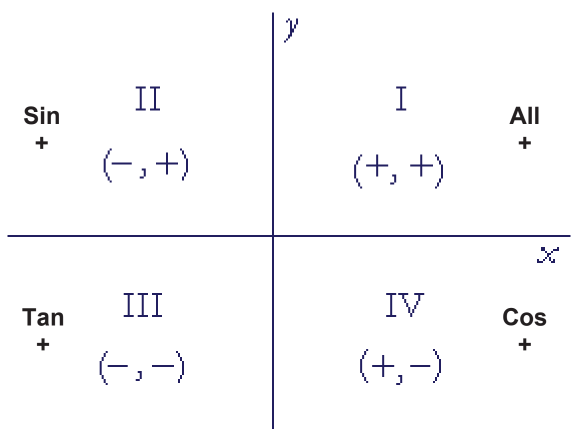
SCO: T3 – Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.
[ME, PS, R, T, V]

Elaboration

Approximate values for trigonometric ratios can be determined using a calculator in the appropriate mode. Exact values of trigonometric ratios for the special angles listed in the table below and their multiples, can be determined using the coordinates of the points on the unit circle.

		sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0^0	**0**	0	1	0	undefined	1	undefined
30^0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45^0	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60^0	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90^0	$\frac{\pi}{2}$	1	0	undefined	1	undefined	0

Given the value of a trigonometric ratio, a calculator may be used to determine one corresponding angle measure. Then, using a knowledge of reference angles, coterminal angles, and signs of ratios in each quadrant, other possible angle measures can be found. Unless the domain is restricted, there will always be an infinite number of answers.



If a point (x, y) lies on the terminal arm of an angle θ in standard position, then the values of the six trigonometric functions of θ can be found using the following formulas:

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{r}{y} \qquad \cot\theta = \frac{x}{y}$$

where $r^2 = x^2 + y^2$.

Grade 12 – Topic: Trigonometry (T)

GCO: Develop trigonometric reasoning.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
T2 Solve problems, using the three primary trigonometric ratios for angles from 0^0 to 360^0 in standard position.	**T4** Graph and analyse the trigonometric functions sine, cosine and tangent to solve problems.

SCO: T4 – Graph and analyse the trigonometric functions sine, cosine and tangent to solve problems. [CN, PS, T, V]

Students who have achieved this outcome should be able to:

A. Sketch, with or without technology, the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$.

B. Determine the characteristics (amplitude, asymptotes, domain, period, range and zeros) of the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$.

C. Determine how varying the value of *a* affects the graphs of $y = a \sin x$ and $y = a \cos x$.

D. Determine how varying the value of *d* affects the graphs of $y = \sin x + d$ and $y = \cos x + d$.

E. Determine how varying the value of *c* affects the graphs of $y = \sin(x + c)$ and $y = \cos(x + c)$.

F. Determine how varying the value of *b* affects the graphs of $y = \sin bx$ and $y = \cos bx$.

G. Sketch, without technology, graphs of the form $y = a \sin b(x - c) + d$ and $y = a \cos b(x - c) + d$, using transformations and explain the strategies.

H. Determine the characteristics (amplitude, asymptotes, domain, period, phase shift, range and zeros) of the graphs of trigonometric functions of the form $y = a \sin b(x - c) + d$ and $y = a \cos b(x - c) + d$.

I. Determine the values of *a*, *b*, *c* and *d* for the functions of the form $y = a \sin b(x - c) + d$ and $y = a \cos b(x - c) + d$ that correspond to a given graph and write the equation of the function.

J. Determine a trigonometric function that models a situation to solve a problem.

K. Explain how the characteristics of the graph of a trigonometric function relate to the conditions in a problem situation.

L. Solve a problem by analysing the graph of a trigonometric function.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

5.1 (A B C F H)

5.2 (C D E F G H I J K L)

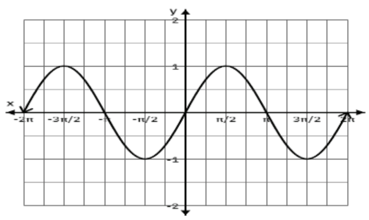
5.3 (A B J K L)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

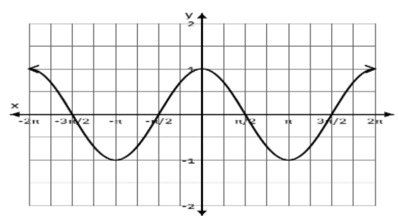
SCO: T4 – Graph and analyse the trigonometric functions sine, cosine and tangent to solve problems.
[CN, PS, T, V]

Elaboration

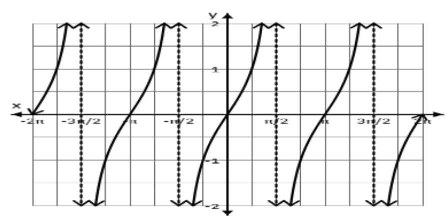
In order to sketch the graphs of $y = \sin x$ and $y = \cos x$, determine the coordinates of the key points representing the *x*-intercepts, maximum points and minimum points. In order to sketch the graph of $y = \tan x$, determine the coordinates of the *x*-intercepts and the equations of the asymptotes. Then to get an accurate graph of each function, choose eight evenly-spaced points in each period of the function, and graph the results.



$y = \sin x$



$y = \cos x$



$y = \tan x$

The following table highlights the characteristics of the graphs of each basic trigonometric function:

	$y = \sin x$	$y = \cos x$	$y = \tan x$
Maximum Value	1	1	none
Minimum Value	–1	–1	none
Amplitude	1	1	none
Period	2π	2π	π
***x*-intercepts**	$\pi n,\ n \in I$	$\frac{\pi}{2} + \pi n,\ n \in I$	$\pi n,\ n \in I$
***y*-intercept**	0	1	0
Domain	$\{x \mid x \in R\}$	$\{x \mid x \in R\}$	$\left\{x \mid x \neq \frac{\pi}{2} + \pi n,\ n \in I\right\}$
Range	$\{y \mid -1 \le y \le 1, y \in \Re\}$	$\{y \mid -1 \le y \le 1, y \in \Re\}$	$\{y \mid y \in R\}$
Vertical Asymptotes	none	none	$x = \frac{\pi}{2} + \pi n,\ n \in I$

The characteristics of sinusoidal functions of the form $y = a \sin b(x - c) + d$ and $y = a \cos b(x - c) + d$ can be summarized as follows:

- The amplitude is represented by $|a|$. It can be found by using the formula $\text{amplitude} = \frac{\max - \min}{2}$.
- The period can be calculated by using the formula $\text{period} = \frac{2\pi}{|b|}$, in radians, or $\text{period} = \frac{360^0}{|b|}$, in degrees.
- The phase shift is represented by *c*. It is a shift to the right if $c > 0$, and to the left if $c < 0$.
- The vertical displacement is represented by *d*. It is up if $d > 0$, and down if $d < 0$.

Grade 12 – Topic: Trigonometry (T)

GCO: Develop trigonometric reasoning.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
T2 Solve problems, using the three primary trigonometric ratios for angles from 0^0 to 360^0 in standard position.	**T5** Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

SCO: **T5 – Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.** **[CN, PS, R, T, V]**

Students who have achieved this outcome should be able to:

A. Verify, with or without technology, that a given value is a solution to a trigonometric equation.

B. Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form when possible.

C. Determine, using technology, the approximate solution of a trigonometric equation in a restricted domain.

D. Relate the general solution of a trigonometric equation to the zeros of the corresponding trigonometric function (restricted to sine and cosine functions).

E. Identify and correct errors in a solution for a trigonometric equation.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

4.4 (A B C D E)

5.4 (A B C)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: T5 – Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]

Elaboration

The same techniques that are used to solve linear and quadratic equations can be used to solve trigonometric equations algebraically. Doing so will result in simplified equations of the form $\sin x = a$, $\cos x = a$, or $\tan x = a$. An initial value of *x* can be found by either using the unit circle for exact values of *x*, or a calculator for approximate values of *x*. Then, reference angles can be used to find other solutions within the given domain.

If a general solution is required for a trigonometric equation, find the solutions in one positive rotation, 2π or 360^0. Then, use the concept of coterminal angles to write an expression that identifies all possible solutions.

If a trigonometric equation simplifies to, or has components that are either of the form $\sin x = a$ or $\cos x = a$, where $a \neq \pm 1$, there will be two sets of general solutions of the form $x + 2\pi n$, or $x + (360n)^0$, $n \in I$, where *x* is an initial solution of the equation. Since the period of $y = \sin x$ and $y = \cos x$ is 2π or 360^0, we add multiples of these values to generate a general solution. If $a = \pm 1$, there will only be one set of general solutions.

As an example, consider the following trigonometric equation:

$$\sqrt{2}\ \sin x = 1$$

$$\sin x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Since $\sin x$ is positive in Quadrants I and II, we have initial values of $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. These initial values can be used to generate the general solutions of $x = \frac{\pi}{4} + 2\pi n$, $n \in I$ and $x = \frac{3\pi}{4} + 2\pi n$, $n \in I$.

If a trigonometric equation simplifies to, or has components that are of the form $\tan x = a$, where *a* is a real number, there will be one set of general solutions of the form $x \pm \pi n$, or $x \pm (180n)^0$, $n \in I$, where *x* is an initial solution of the equation. Since the period of $y = \tan x$ is π or 180^0, we add multiples of the appropriate value to generate a general solution.

The solution to a trigonometric equation can also be determined and illustrated using graphing software.

Grade 12 – Topic: Trigonometry (T)

GCO: Develop trigonometric reasoning.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	T6 Prove trigonometric identities, using: • reciprocal identities; • quotient identities; • Pythagorean identities; • sum or difference identities (restricted to sine, cosine and tangent); • double-angle identities (restricted to sine, cosine and tangent).

SCO: **T6 – Prove trigonometric identities, using:**

- **reciprocal identities;**
- **quotient identities;**
- **Pythagorean identities;**
- **sum or difference identities (restricted to sine, cosine and tangent);**
- **double-angle identities (restricted to sine, cosine and tangent).**

[R, T, V]

Students who have achieved this outcome should be able to:

A. Explain the difference between a trigonometric identity and a trigonometric equation.

B. Verify a trigonometric identify numerically for a given value in either degrees or radians.

C. Explain why verifying that the two sides of a trigonometric identity are equal for given values is insufficient to conclude that the identity is valid.

D. Determine, graphically, the potential validity of a trigonometric identity, using technology.

E. Prove, algebraically, that a trigonometric identity is valid.

F. Determine, using the sum, difference and double-angle identities, the exact value of a trigonometric ratio.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

6.1 (A B C D)

6.2 (B D F)

6.3 (C E F)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: T6 – Prove trigonometric identities, using:
- **reciprocal identities;**
- **quotient identities;**
- **Pythagorean identities;**
- **sum or difference identities (restricted to sine, cosine and tangent);**
- **double-angle identities (restricted to sine, cosine and tangent).**

[R, T, V]

Elaboration

A trigonometric identity is an equation involving trigonometric functions that is true for all permissible values of the variable. While non-permissible values should be discussed, it is not the intent of this outcome to engage students in identifying non-permissible values. Trigonometric identities can be verified numerically and graphically for different values of the variable, but this is not sufficient to conclude that an equation is an identity. This can only be done through algebraic proof. Also, trigonometric expressions can be used to simplify more complicated trigonometric expressions and determine exact trigonometric values for some angles.

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

Sum Identities

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Difference Identities

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double-Angle Identities

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

To prove a trigonometric identity algebraically, rewrite both sides of the identity separately into identical expressions. Do not move terms from one side of the identity to the other. It is usually easier to make a more complicated expression simpler than to make a simple expression more complicated. In no particular order, common strategies that can be used when proving identities are:

- Use known quantities to make substitutions.
- If quadratics are present, the Pythagorean identity, or one of its alternate forms, can often be used.
- Rewrite each expression using sine and cosine only.
- Multiply the numerator and the denominator by the conjugate of an expression.
- Factor to simplify expressions.

RELATIONS AND FUNCTIONS

SPECIFIC CURRICULUM OUTCOMES

RF1 – Demonstrate an understanding of operations on, and compositions of, functions.

RF2 – Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

RF3 – Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

RF4 – Apply translations and stretches to the graphs and equations of functions.

RF5 – Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:

- ***x*-axis;**
- ***y*-axis;**
- **line $y = x$.**

RF6 – Demonstrate an understanding of inverses of relations.

RF7 – Demonstrate an understanding of logarithms.

RF8 – Demonstrate an understanding of the product, quotient and power laws of logarithms.

RF9 – Graph and analyse exponential and logarithmic functions.

RF10 – Solve problems that involve exponential and logarithmic equations.

RF11 – Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree $\leq$ 5 with integral coefficients).

RF12 – Graph and analyse polynomial functions (limited to polynomials functions of degree $\leq$ 5).

RF13 – Graph and analyse radical functions (limited to functions involving one radical).

RF14 – Graph and analyse rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic reasoning and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	RF1 Demonstrate an understanding of operations on, and compositions of, functions.

SCO: **RF1 – Demonstrate an understanding of operations on, and compositions of, functions.** **[CN, R, T, V]**

Students who have achieved this outcome should be able to:

A. Sketch the graph of a function that is the sum, difference, product or quotient of two functions, given their graphs.

B. Write the equations of a function that is the sum, difference, product or quotient of two or more functions, given their equations.

C. Determine the domain and range of a function that is the sum, difference, product or quotient of two functions.

D. Write a function $h(x)$ as the sum, difference, product or quotient of two or more functions.

E. Determine the value of the composition of functions when evaluated at a point, including:
- $f[f(a)]$;
- $f[g(a)]$;
- $g[f(a)]$.

F. Determine, given the equations of two functions $f(x)$ and $g(x)$, the equations of the composite functions:
- $f[f(x)]$;
- $f[g(x)]$;
- $g[f(x)]$;

and explain any restrictions.

G. Sketch, given the equations of two functions $f(x)$ and $g(x)$, the graph of the composite functions:
- $f[f(x)]$;
- $f[g(x)]$;
- $g[f(x)]$.

H. Write a function $h(x)$ as the composition of two or more functions.

I. Write a function $h(x)$ by combining two or more functions through operations on, and composition of, functions.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

10.1 (A B C D) **10.2 (A B C D)**

10.3 (E F G H I)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: RF1 – Demonstrate an understanding of operations on, and compositions of, functions. [CN, R, T, V]

Elaboration

Two functions, $f(x)$ and $g(x)$, can be combined in one of five ways, as described below:

OPERATION	FUNCTION NOTATION	DOMAIN
Addition	$(f+g)(x) = f(x) + g(x)$	Common to $f(x)$ and $g(x)$
Subtraction	$(f-g)(x) = f(x) - g(x)$	Common to $f(x)$ and $g(x)$
Multiplication	$(f \bullet g)(x) = f(x) \bullet g(x)$	Common to $f(x)$ and $g(x)$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Common to $f(x)$ and $g(x)$, where $g(x) \neq 0$
Composition	$(f \circ g)(x) = f\left[g(x)\right]$	Common to $(f \circ g)(x)$ and $g(x)$

The range of each of these function combinations can be determined using its graph. To sketch the graph of the sum, difference, product, or quotient of two functions, perform the indicated operation on the *y*-coordinates of the respective points for each *x*-coordinate. Then develop a table of values using this new set of points in order to draw a sketch of its graph.

Please note that the operation of composition is not commutative. For example, consider the functions $f(x) = x+1$ and $g(x) = x^2$:

$$(f \circ g)(x) = f\left[g(x)\right] = f\left(x^2\right) = x^2 + 1$$

$$(g \circ f)(x) = g\left[f(x)\right] = g(x+1) = (x+1)^2 = x^2 + 2x + 1$$

More time should be spent on the composition of functions so teachers need to be selective in the quantity of exercises completed for the operation of functions.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic reasoning and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	RF2 Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

SCO: **RF2 – Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.** [C, CN, R, V]

Students who have achieved this outcome should be able to:

A. Compare the graphs of a set of functions of the form $y-k=f(x)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of *k*.

B. Compare the graphs of a set of functions of the form $y=f(x-h)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of *h*.

C. Compare the graphs of a set of functions of the form $y-k=f(x-h)$, to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effects of *h* and *k*.

D. Sketch the graph of $y-k=f(x)$, $y=f(x-h)$ or $y-k=f(x-h)$ for given values of *h* and *k*, given a sketch of the function $y=f(x)$, where the equation of $y=f(x)$ is not given.

E. Write the equation of a function whose graph is a vertical and/or horizontal translation of the graph of the function $y=f(x)$.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

1.1 (A B C D E)

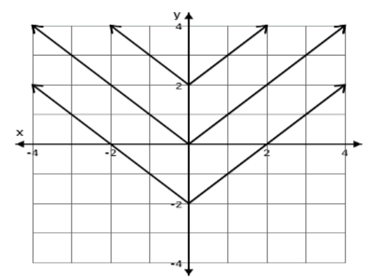
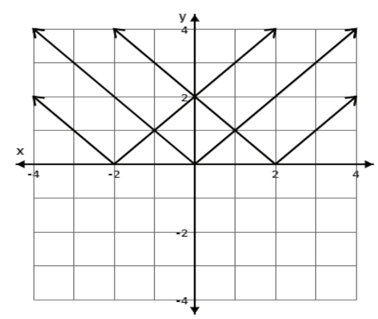
2.1 (A B)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: RF2 – Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations. [C, CN, R, V]

Elaboration

Translations are transformations that shift all points on the graph of a function up, down, left, or right without changing the shape or the orientation of the graph. A sketch of the graph of $y-k=f(x-h)$, often rewritten as $y=f(x-h)+k$, can be created by translating key points on the graph of the base function $y=f(x)$. The table summarizes the translations of the function $y=f(x)$.

FUNCTION	TRANSFORMATION FROM $y=f(x)$	MAPPING	EXAMPLE
$y-k=f(x)$, or $y=f(x)+k$	A vertical translation: If $k>0$, the translation is up. If $k<0$, the translation is down.	$(x,y)\rightarrow(x,y+k)$	
$y=f(x-h)$	A horizontal translation: If $h>0$, the translation is to the right. If $h<0$, the translation is to the left.	$(x,y)\rightarrow(x+h,y)$	

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic reasoning and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
RF3 Analyse quadratic functions of the form $y = a(x-p)^2 + q$ and determine the: • vertex; • domain and range; • direction of opening; • axis of symmetry; • *x*- and *y*-intercepts.	**RF3** Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

SCO: **RF3 – Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.** **[C, CN, R, V]**

Students who have achieved this outcome should be able to:

A. Compare the graphs of a set of functions of the form $y = af(x)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of *a*.

B. Compare the graphs of a set of functions of the form $y = f(bx)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of *b*.

C. Compare the graphs of a set of functions of the form $y = af(bx)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effects of *a* and *b*.

D. Sketch the graph of $y = af(x)$, $y = f(bx)$ or $y = af(bx)$ for given values of *a* and *b*, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

E. Write the equation of a function, given its graph which is a vertical and/or horizontal stretch of the graph of the function $y = f(x)$.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

1.2 (A B C D E)

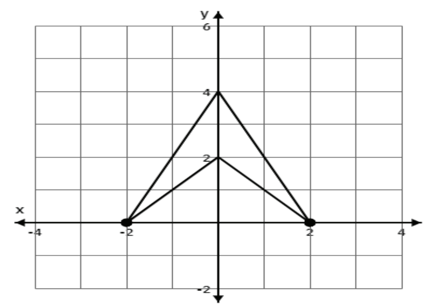
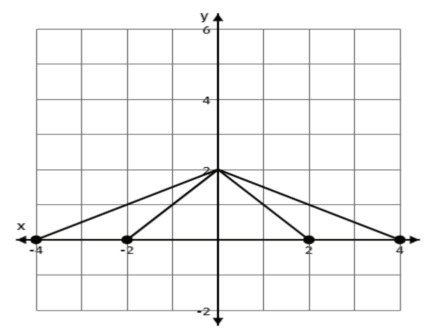
2.1 (A B)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: RF3 – Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. [C, CN, R, V]

Elaboration

Stretches are transformations that change the shape of a graph by a constant scale factor. If the scale factor is greater than 1, the resulting image is larger than the original graph. If the scale factor is between 0 and 1, the resulting image is smaller than the original graph. A sketch of the graph of a stretch can be created by translating key points on the graph of the base function $y = f(x)$. The table summarizes the stretches of the function $y = f(x)$.

FUNCTION	TRANSFORMATION FROM $y = f(x)$	MAPPING	EXAMPLE
$y = af(x)$	A vertical stretch about the *x*-axis by a factor of $\|a\|$; if $a < 0$, then the graph is also reflected in the *x*-axis	$(x, y) \rightarrow (x, ay)$	
$y = f(bx)$	A horizontal stretch about the *y*-axis by a factor of $\frac{1}{\|b\|}$; if $b < 0$, then the graph is also reflected in the *y*-axis	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic reasoning and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
RF3 Analyse quadratic functions of the form $y = a(x-p)^2 + q$ and determine the: • vertex; • domain and range; • direction of opening; • axis of symmetry; • *x*- and *y*-intercepts.	**RF4** Apply translations and stretches to the graphs and equations of functions.

SCO: **RF4 – Apply translations and stretches to the graphs and equations of functions.** [C, CN, R, V]

Students who have achieved this outcome should be able to:

A. Sketch the graph of the function $y - k = af\left[b(x-h)\right]$ for given values of *a*, *b*, *h* and *k*, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

B. Write the equation of a function, given its graph which is a translation and/or stretch of the graph of the function $y = f(x)$.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

1.3 (A B)

2.1 (A B)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: RF4 – Apply translations and stretches to the graphs and equations of functions. [C, CN, R, V]

Elaboration

When combining transformations, the transformed function should be written in the form $y = af\left[b\left(x-h\right)\right]+k$ in order to better identify the individual transformations. Note that the order in which transformations are performed is important. Stretches and reflections may be performed in any order, but are always performed before translations. This corresponds to the order of operations on real numbers.

When the graph of the transformed function $y = af\left[b\left(x-h\right)\right]+k$ is compared to the graph of $y = f\left(x\right)$, the parameters *a*, *b*, *h*, and *k*, correspond to the following transformations:

- *a* corresponds to a vertical stretch about the *x*-axis by a factor of $|a|$; if $a < 0$, then the function is also reflected in the *x*-axis
- *b* corresponds to a horizontal stretch about the *y*-axis by a factor of $\frac{1}{|b|}$; if $b < 0$, then the function is also reflected in the *y*-axis
- *h* corresponds to a horizontal translation
- *k* corresponds to a vertical translation

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic reasoning and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
RF11 Graph and analyse reciprocal functions (limited to the reciprocal of linear functions).	**RF5** Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the: • x-axis; • y-axis; • line $y = x$.

SCO: **RF5 – Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:**

- ***x*-axis;**
- ***y*-axis;**
- **line $y = x$.**

[C, CN, R, V]

Students who have achieved this outcome should be able to:

A. Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corresponding ordered pair that results from a reflection through the x-axis, the y-axis or the line $y = x$.

B. Sketch the reflection of the graph of a function $y = f(x)$ through the x-axis, the y-axis or the line $y = x$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

C. Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function $y = f(x)$ through the x-axis, the y-axis or the line $y = x$.

D. Sketch the graphs of the functions $y = -f(x)$, $y = f(-x)$ and $x = f(y)$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

E. Write the equation of a function, given its graph which is a reflection of the graph of the function $y = f(x)$ through the x-axis, the y-axis or the line $y = x$.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

1.2 (A B C D E)

1.4 (A B C D E)

2.1 (A B)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

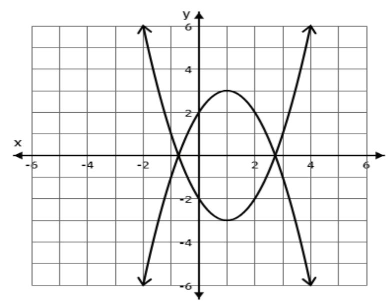
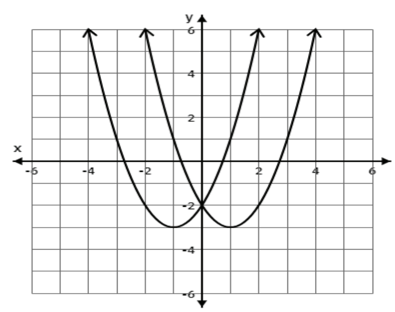
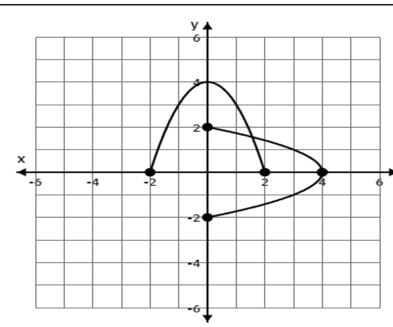
SCO: RF5 – Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:

- ***x*-axis;**
- ***y*-axis;**
- **line $y = x$.**

[C, CN, R, V]

Elaboration

Reflections are transformations that reflect a graph through a line without changing the shape of the graph. A sketch of the graph of a reflection can be created by translating key points on the graph of the base function $y = f(x)$. The table summarizes the reflections of the function $y = f(x)$.

FUNCTION OR RELATION	TRANSFORMATION FROM $y = f(x)$	MAPPING	EXAMPLE
$y = -f(x)$	A reflection in the *x*-axis	$(x, y) \rightarrow (x, -y)$	
$y = f(-x)$	A reflection in the *y*-axis	$(x, y) \rightarrow (-x, y)$	
$x = f(y)$	A reflection in the line $y = x$	$(x, y) \rightarrow (y, x)$	

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic reasoning and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
RF11 Graph and analyse reciprocal functions (limited to the reciprocal of linear functions).	**RF6** Demonstrate an understanding of inverses of relations.

SCO: **RF6 – Demonstrate an understanding of inverses of relations.** **[C, CN, R, V]**

Students who have achieved this outcome should be able to:

A. Explain how the graph of the line $y = x$ can be used to sketch the inverse of a relation.

B. Explain how the transformation $(x, y) \rightarrow (y, x)$ can be used to sketch the inverse of a relation.

C. Sketch the graph of the inverse relation, given the graph of a relation.

D. Determine if a relation and its inverse are functions.

E. Determine restrictions on the domain of a function in order for its inverse to be a function.

F. Determine the equation and sketch the graph of the inverse relation, given the equation of a linear or quadratic relation.

G. Explain the relationship between the domains and ranges of a relation and its inverse.

H. Determine, algebraically or graphically, if two functions are inverses of each other.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

1.4 (A B C D E F G H)

2.1 (A B)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: RF6 – Demonstrate an understanding of inverses of relations. [C, CN, R, V]

Elaboration

The inverse of a relation can be found by interchanging the *x*-coordinates and *y*-coordinates of the points on its graph. The graph of the inverse of a relation is the graph of the relation reflected in the line $y = x$. As a result, it is easy to verify graphically that two relations are inverses of each other. The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.

Not all functions have inverses that are functions. The horizontal line test can be used to determine if the inverse will be a function. However, an inverse can be created that is a function by restricting the domain of the original function to a specified interval. When the inverse of a function, $f(x)$, is itself a function, it is denoted by $f^{-1}(x)$. However, students must be careful when using this notation, as it is not equal to $\frac{1}{f(x)}$.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic reasoning and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	RF7 Demonstrate an understanding of logarithms.

SCO: **RF7 – Demonstrate an understanding of logarithms.** [CN, ME, R]

Students who have achieved this outcome should be able to:

A. Explain the relationship between logarithms and exponents.

B. Express a logarithmic expression as an exponential expression and vice versa.

C. Determine, without technology, the exact value of a logarithm, such as $\log_2 8$.

D. Estimate the value of a logarithm, using benchmarks, and explain the reasoning, *e.g.,* since $\log_2 8 = 3$ and $\log_2 16 = 4$, $\log_2 9$ is approximately equal to 3.1.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

8.1 (A B C D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: RF7 – Demonstrate an understanding of logarithms. [CN, ME, R]

Elaboration

Evaluating a logarithm is equivalent to finding an exponent. As a result, equations in exponential form can be written in logarithmic form, and vice versa. The exponential equation $x = c^y$ is equivalent to the logarithmic equation $y = \log_c x$. In both equations, *c* is the base and *y* is the exponent.

The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$, which is equivalent to $y = \log_c x$, written in logarithmic form. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$, which is equivalent to $y = c^x$, written in exponential form.

A common logarithm has base 10. It is not necessary to write the base for common logarithms, so $\log_{10} x$ can be written as $\log x$.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	RF8 Demonstrate an understanding of the product, quotient and power laws of logarithms.

SCO: **RF8 – Demonstrate an understanding of the product, quotient and power laws of logarithms.** **[C, CN, R, T]**

Students who have achieved this outcome should be able to:

A. Develop and generalize the laws for logarithms, using numeric examples and exponent laws.

B. Determine, using the laws of logarithms, an equivalent expression for a logarithmic expression.

C. Determine, with technology, the approximate value of a logarithmic expression, such as $\log_2 9$.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

8.3 (A B)

8.4 (C)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: RF8 – Demonstrate an understanding of the product, quotient and power laws of logarithms. [C, CN, R, T]

Elaboration

Logarithmic expressions involving multiplication, division, and exponents can be rewritten by using the laws of logarithms. Let *P* be any real number, and *M*, *N*, and *c* be positive real numbers, where $c \neq 1$. Then, the following laws of logarithms are valid:

NAME	LAW	DESCRIPTION
Product Rule	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient Rule	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
Power Rule	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the power times the logarithm of the number.

The laws of logarithms are to be derived to demonstrate the laws apply in all contexts.

In order to evaluate a logarithm such as $\log_2 9$, where the argument is not a rational power of the base, we can do the following:

Let $x = \log_2 9$. Rewriting this equation as an exponential equation gives us $2^x = 9$. Now, if we take the common logarithm of both sides, we get $\log 2^x = \log 9$. Applying the Power Rule of Logarithms converts this equation to $x \log 2 = \log 9$, and solving for *x* leads us to $x = \frac{\log 9}{\log 2} \approx 3.1699$.

Using this method gives us the following general result:

$$\log_c M = \frac{\log M}{\log c}$$

This is called the Change of Base Formula for Logarithms.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	RF9 Graph and analyse exponential and logarithmic functions.

SCO: **RF9 – Graph and analyse exponential and logarithmic functions.** **[C, CN, T, V]**

Students who have achieved this outcome should be able to:

A. Sketch, with or without technology, the graph of an exponential function of the form $y = a^x$, $a > 0$.

B. Identify the characteristics of the graph of an exponential function of the form $y = a^x$, $a > 0$, including the domain, range, horizontal asymptote and intercepts, and explain the significance of the horizontal asymptote.

C. Sketch the graph of an exponential function by applying a set of transformations to the graph of $y = a^x$, $a > 0$. and state the characteristics of the graph.

D. Sketch, with or without technology, the graph of a logarithmic function of the form $y = \log_b x$, $b > 1$.

E. Identify the characteristics of the graph of a logarithmic function of the form $y = \log_b x$, $b > 1$, including the domain, range, vertical asymptote and intercepts, and explain the significance of the vertical asymptote.

F. Sketch the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_b x$, $b > 1$, and state the characteristics of the graph.

G. Demonstrate, graphically, that a logarithmic function and an exponential function with the same base are inverses of each other.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

7.1 (A B)

7.2 (C)

8.1 (D E G)

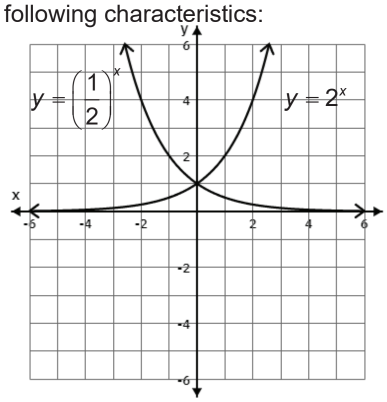
8.2 (F)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: RF9 – Graph and analyse exponential and logarithmic functions. [C, CN, T, V]

Elaboration

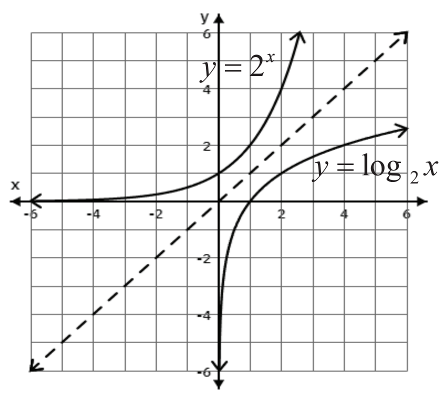
An exponential function of the form $y = c^x$, $c > 0$, has the following characteristics:



- is increasing if $c > 1$
- is decreasing if $0 < c < 1$
- is a horizontal line if $c = 1$
- has a domain of $\{x \mid x \in R\}$
- has a range of $\{y \mid y > 0\}$ has a y-intercept of 1
- has no x-intercept
- has a horizontal asymptote at $x = 0$

To sketch the graph of an exponential function of the form $y = a(c)^{b(x-h)} + k$, apply transformations to the graph of $y = c^x$, where $c > 0$. Remember that the transformations represented by a and b are applied before the transformations represented by h and k.

The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown below right. A logarithmic function of the form $y = \log_c x$, $c > 0$, $c \neq 1$, has the following characteristics:



- is increasing if $c > 1$
- is decreasing if $0 < c < 1$
- has a domain of $\{x \mid x > 0\}$
- has a range of $\{y \mid y \in R\}$
- has an x-intercept of 1
- has no y-intercept
- has a vertical asymptote at $y = 0$

To sketch the graph of a logarithmic function of the form $y = a \log_c [b(x-h)] + k$, apply transformations to the graph of $y = \log_c x$, where $c > 0$. Remember that the transformations represented by a and b are applied before the transformations represented by h and k.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	RF10 Solve problems that involve exponential and logarithmic equations.

SCO: **RF10 – Solve problems that involve exponential and logarithmic equations.** **[C, CN, PS, R]**

Students who have achieved this outcome should be able to:

A. Determine the solution of an exponential equation in which the bases are powers of one another.

B. Determine the solution of an exponential equation in which the bases are not powers of one another, using a variety of strategies.

C. Determine the solution of a logarithmic equation and verify the solution.

D. Solve a problem that involves exponential growth or decay.

E. Solve a problem that involves the application of exponential equations to loans, mortgages and investments.

F. Solve a problem that involves logarithmic scales, such as the Richter scale and the pH scale.

G. Solve a problem by modelling a situation with an exponential or a logarithmic equation.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

7.1 (D)

7.2 (D)

7.3 (A D E)

8.3 (F)

8.4 (B C D E G)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: RF10 – Solve problems that involve exponential and logarithmic equations. [C, CN, PS, R]

Elaboration

Some exponential equations can be solved directly if the terms on either side of the equal sign have the same base or can be rewritten so they have the same base.

- If the bases are the same, then equate the exponents and solve for the variable.
- If the bases are different but can be rewritten with the same base, use the exponent laws, and then equate the exponents and solve for the variable.

When solving a logarithmic equation algebraically, start by applying the laws of logarithms to express each side of the equation as a single logarithm, then equate the arguments and solve for the variable. Following are four useful properties when solving logarithmic equations. In all cases, c, L, $r > 0$ and $c \neq 1$.

- If $\log_c L = \log_c R$, then $L = R$.
- The equation $\log_c L = R$ can be rewritten with logarithms on both sides of the equation as $\log_c L = \log_c c^R$.
- The equation $\log_c L = R$ can be written in exponential form as $L = c^R$.
- The logarithm of zero or a negative number is undefined. To identify whether a possible root is extraneous, always substitute it back into the original equation and check whether all of the logarithms are defined.

All exponential equations can be solved algebraically by taking logarithms of both sides of the equation. Then apply the Power Rule for Logarithms to solve for the variable.

Many real-world situations, such as population growth and radioactive decay, can be modelled with an exponential or a logarithmic equation. A general model for many problems involving exponential growth or decay is

$$\text{final quantity} = (\text{initial quantity}) \bullet (\text{change factor})^{(\text{number of changes})}$$

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
RF1 Factor polynomials of the form: • $ax^2 + bx + c$, $a \neq 0$; • $a^2x^2 - b^2y^2$, $a \neq 0$, $b \neq 0$; where *a*, *b* and *c* are rational numbers.	**RF11** Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree $\leq$ 5 with integral coefficients).

SCO: **RF11 – Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree $\leq$ 5 with integral coefficients). [C, CN, ME]**

Students who have achieved this outcome should be able to:

A. Explain how long division of a polynomial expression by a binomial expression of the form $x - a$, $a \in I$, is related to synthetic division.

B. Divide a polynomial expression by a binomial expression of the form $x - a$, $a \in I$, using long division or synthetic division.

C. Explain the relationship between the linear factors of a polynomial expression and the zeros of the corresponding polynomial function.

D. Explain the relationship between the remainder when a polynomial expression is divided by $x - a$, $a \in I$, and the value of the polynomial expression at $x = a$ (remainder theorem).

E. Explain and apply the factor theorem to express a polynomial expression as a product of factors.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

3.2 (A B D)

3.3 (C E)

[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: RF11 – Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree $\leq$ 5 with integral coefficients). [C, CN, ME]

Elaboration

A polynomial can be divided by a binomial by using either long division or synthetic division. The result of the division of a polynomial in *x*, $P(x)$, by a binomial of the form $x-a$ can be written as $\frac{P(x)}{x-a}=Q(x)+\frac{R}{x-a}$ or $P(x)=(x-a)Q(x)+R$, where $Q(x)$ is the quotient and *R* is the remainder.

To check the result of a division, multiply the quotient, $Q(x)$, by the divisor, $x-a$, and add the remainder, *R*, to the product. The result should be the dividend, $P(x)$.

The remainder theorem states that when a polynomial in *x*, $P(x)$, is divided by a binomial of the form $x-a$, the remainder is $P(a)$. A non-zero remainder means that the binomial is not a factor of $P(x)$. The factor theorem states that $x-a$ is a factor of a polynomial $P(x)$ if and only if $P(a)=0$. The integral zero theorem states that if $x-a$ is a factor of the polynomial function $P(x)$ with integral coefficients, then *a* is a factor of the constant term of $P(x)$.

The factor theorem and the integral zero theorem can be used to factor some polynomial functions, using the following procedure:

- Use the integral zero theorem to list possible integer values for the zeroes.
- Next, apply the factor theorem to determine one factor.
- Then, use division to determine the missing factor.
- Repeat the above steps until all factors are found, the result is a quadratic that can be factored using traditional strategies, or the remaining factor cannot be factored.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
RF4 Analyse quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: • vertex; • domain and range; • direction of opening; • axis of symmetry; • *x*- and *y*-intercepts and to solve problems.	**RF12** Graph and analyse polynomial functions (limited to polynomials functions of degree $\leq$ 5).

SCO: **RF12 – Graph and analyse polynomial functions (limited to polynomials functions of degree $\leq$ 5).**
[C, CN, T, V]

Students who have achieved this outcome should be able to:

A. Identify the polynomial functions in a set of functions, and explain the reasoning.

B. Explain the role of the constant term and leading coefficient in the equation of a polynomial function with respect to the graph of the function.

C. Generalize rules for graphing polynomial functions of odd or even degree.

D. Explain the relationship between:
- the zeros of a polynomial function;
- the roots of the corresponding polynomial equation;
- the *x*-intercepts of the graph of the polynomial function.

E. Explain how the multiplicity of a zero of a polynomial function affects the graph.

F. Sketch, with or without technology, the graph of a polynomial function.

G. Solve a problem by modelling a given situation with a polynomial function and analysing the graph of the function.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

3.1 (A B C)

3.4 (D E F G)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

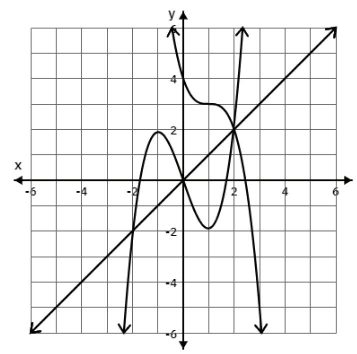
SCO: RF12 – Graph and analyse polynomial functions (limited to polynomials functions of degree $\le$ 5). [C, CN, T, V]

Elaboration

A polynomial function has the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_2 x^2 + a_1 x + a_0$, where a_n is the leading coefficient, a_0 is the constant, and the degree of the polynomial, *n*, is the exponent of the greatest power of the variable, *x*.

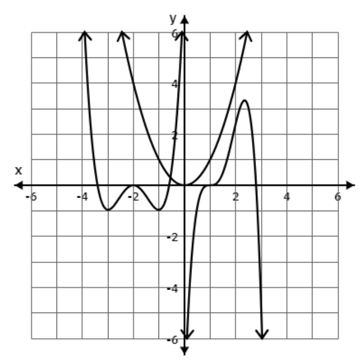
Graphs of odd degree have the following characteristics:

- a graph that extends down into Quadrant III and up into Quadrant I when the leading coefficient is positive
- a graph that extends up into Quadrant II and down into Quadrant IV when the leading coefficient is negative
- a *y*-intercept that corresponds to the constant term of the function
- at least one *x*-intercept and up to a maximum of *n* *x*-intercepts, where *n* is the degree of the function
- a domain of $\{x \mid x \in R\}$ and a range of $\{y \mid y \in R\}$
- no maximum or minimum points



Graphs of even degree have the following characteristics:

- a graph that extends up into Quadrant II and up into Quadrant I when the leading coefficient is positive
- a graph that extends down into Quadrant III and down into Quadrant IV when the leading coefficient is negative
- a *y*-intercept that corresponds to the constant term of the function
- from zero to a maximum of *n* *x*-intercepts, where *n* is the degree of the function
- a domain of $\{x \mid x \in R\}$ and a range that depends on the maximum or minimum value of the function



The graph of a polynomial function can be sketched by using the intercepts, the degree of the function, and the sign of the leading coefficient. When a polynomial is in factored form with a factor repeated *n* times, the corresponding zero has multiplicity *n*. The shape of a graph close to a zero of $x = a$ (multiplicity *n*) is similar to the shape of the graph of a function with degree equal to *n* of the form $y = (x - a)^n$. Polynomial functions change sign at *x*-intercepts that correspond to zeros of odd multiplicity. The graph crosses over the *x*-axis at these intercepts. Polynomial functions do not change sign at *x*-intercepts that correspond to zeros of even multiplicity. The graph touches, but does not cross, the *x*-axis at these intercepts.

To sketch the graph of a polynomial function of the form $y = a\left[b(x - h)\right]^n + k$, apply transformations to the graph of $y = x^n$, where $n \in N$. Remember that the transformations represented by *a* and *b* are applied before the transformations represented by *h* and *k*.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
AN3 Solve radical equations (limited to square roots).	**RF13** Graph and analyse radical functions (limited to functions involving one radical).

SCO: **RF13 – Graph and analyse radical functions (limited to functions involving one radical).** **[CN, R, T, V]**

Students who have achieved this outcome should be able to:

A. Sketch the graph of the function of $y = \sqrt{x}$, using a table of values, and state the domain and range.

B. Sketch the graph of the function $y - k = a\sqrt{b(x-h)}$ by applying transformations to the graph of the function $y = \sqrt{x}$, and state the domain and range.

Note: It is intended that this outcome be integrated throughout the study of transformations in chapter 1.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

2.1 (A B)

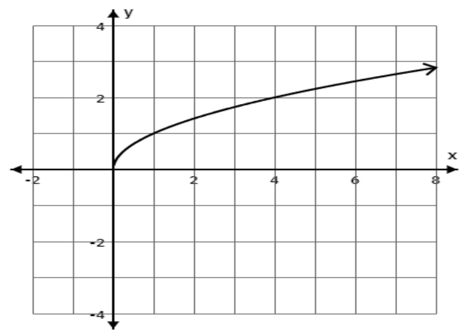
[C] Communication **[CN]** Connections	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving **[R]** Reasoning	**[T]** Technology **[V]** Visualization

SCO: RF13 – Graph and analyse radical functions (limited to functions involving one radical). [CN, R, T, V]

Elaboration

The radical function $y = \sqrt{x}$ has the following characteristics:

- has a left endpoint at $(0,0)$
- has no right endpoint
- is the shape of half a parabola
- has a domain of $\{x \mid x \geq 0\}$
- has a range of $\{y \mid y \geq 0\}$



To sketch the graph of a radical function of the form $y = a\sqrt{b(x-h)} + k$, apply transformations to the graph of $y = \sqrt{x}$. Remember that the transformations represented by *a* and *b* are applied before the transformations represented by *h* and *k*.

Radical function transformations are to be integrated simultaneously with the other function transformation outcomes.

MAT621B – Topic: Relations and Functions (RF)

GCO: Develop algebraic and graphical reasoning through the study of relations.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
AN6 Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).	**RF14** Graph and analyse rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).

SCO: **RF14 – Graph and analyse rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]**

Students who have achieved this outcome should be able to:

A. Graph, with or without technology, a rational function.

B. Analyse the graphs of a set of rational functions to identify common characteristics.

C. Explain the behaviour of the graph of a rational function for values of the variable near a non-permissible value.

D. Determine if the graph of a rational function will have an asymptote or a hole for a non-permissible value.

E. Match a set of rational functions to their graphs, and explain the reasoning.

F. Describe the relationship between the roots of a rational equation and the *x*-intercepts of the graph of the corresponding rational function.

G. Determine, graphically, an approximate solution of a rational equation.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

9.1 (A B C)

9.2 (A C D E)

9.3 (F G)

[C] Communication	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving	**[T]** Technology
[CN] Connections		**[R]** Reasoning	**[V]** Visualization

SCO: RF14 – Graph and analyse rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]

Elaboration

Rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x) \neq 0$. Rational functions where $p(x)$ and $q(x)$ have no common factor other than one have vertical asymptotes that correspond to the non-permissible values of the function, if there are any. Transformations can be used to graph rational functions and to explain common characteristics and differences between them, Equations of some rational functions can be expressed in an equivalent form and can be used to analyse and graph functions without using technology.

The graph of a rational function

- has either a vertical asymptote or a point of discontinuity corresponding to each of its non-permissible values;
- has no vertical asymptotes nor points of discontinuity if there are no non-permissible values.

To find any *x*-intercepts, points of discontinuity, or vertical asymptotes of a rational function, analyse the numerator and the denominator.

- A factor of only the numerator corresponds to an *x*-intercept.
- A factor of only the denominator corresponds to a vertical asymptote.
- A factor of both the numerator and the denominator corresponds to a point of discontinuity.

To analyse the behaviour of a function near a non-permissible value, use a table of values using *x*-values very close to the non-permissible value.

Rational equations can be solved both algebraically and graphically. The solutions or roots of a rational equation are equivalent to the *x*-intercepts of the graph of the corresponding rational function. Either of the following methods can be used to solve rational equations graphically:

- Manipulate the equation so that one side is equal to zero; then, graph the corresponding function and identify the value(s) of the *x*-intercepts.
- Graph the system of functions that corresponds to the expression on each side of the equal sign, and then identify the value(s) of *x* at the point(s) of intersection.

When solving rational equations, always remember to check for extraneous roots and to verify that the solution does not include any non-permissible values in the original equation.

PERMUTATIONS, COMBINATIONS AND THE BINOMIAL THEOREM

SPECIFIC CURRICULUM OUTCOMES

PC1 – Solve problems involving the fundamental counting principle, permutations, and combinations.

PC2 – Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

MAT621B – Topic: Permutations, Combinations and the Binomial Theorem (C)

GCO: Develop algebraic and numeric reasoning that involves combinatorics.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	PC1 Solve problems involving the fundamental counting principle, permutations, and combinations.

SCO: **PC1 - Solve problems involving the fundamental counting principle, permutations, and combinations.**
[C, PS, R, V]

Students who have achieved this outcome should be able to:

A. Count the total number of possible choices that can be made using graphic organizers such as lists and tree diagrams.

B. Solve a simple counting problem by applying the fundamental counting principle.

C. Determine, in factorial notation, the number of permutations of n different elements taken n at a time to solve a problem.

D. Determine, using a variety of strategies, the number of permutations of n different elements taken r at a time to solve a problem.

E. Explain, using examples, the difference between a permutation and a combination.

F. Determine the number of combinations of n different elements taken r at a time to solve a problem.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

11.1 (A BC D)

11.2 (E F)

[C] Communication	**[ME]** Mental Mathematics and Estimation	**[PS]** Problem Solving	**[T]** Technology
[CN] Connections		**[R]** Reasoning	**[V]** Visualization

SCO: PC1 – Solve problems involving the fundamental counting principle, permutations, and combinations. [C, PS, R, V]

Elaboration

Combinatorics refers to the branch of mathematics that deals with counting the number of objects in a set which have a particular pattern or characteristic.

The fundamental counting principle can be used to determine the number of different arrangements made up of a number of tasks. If one task can be performed in *a* ways, a second task in *b* ways, and a third task in *c* ways, and so on, then all tasks can be arranged in $a \bullet b \bullet c \bullet \cdots$ ways. Graphic organizers, such as tables, lists, or tree diagrams can be used to list all of these arrangements. For example, the standard Prince Edward Island license plate consists of 2 letters followed by 3 digits. To calculate the total number of license plates, we determine how many possible characters there can be in each of the 5 positions of a license plate. Since there are 26 letters in the alphabet and 10 digits in the decimal system, we can say that there is a total of $26 \times 26 \times 10 \times 10 \times 10$, or 676,000 possible Prince Edward Island license plates.

Factorial notation, denoted by n!, is often used in combinatorics. It is used for products of successive positive integers. The expression n! is defined as

$$n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

for any positive value of *n*. By convention, 0! is defined as 1.

A permutation is an arrangement of distinct objects in a definite order. The number of permutations of *n* different objects taken *r* at a time is given by

$${}_nP_r = \frac{n!}{(n-r)!}$$

As a consequence, a permutation of *n* distinct objects where all of the objects are included in the arrangement is ${}_nP_r$ or n!.

A combination is a selection of objects in which order is not important. This is the major difference between a permutation problem and a combination problem. When determining the number of possibilities in a situation, if order matters, it is a permutation problem. If order does not matter, it is a combination problem.

The number of combinations of *n* different objects taken *r* at a time can be represented by ${}_nC_r$ where $n \geq r$ and $r \geq 0$. The formula for ${}_nC_r$ is

$${}_nC_r = \frac{{}_nP_r}{r!} \ or \ {}_nC_r = \frac{n!}{r!(n-r)!}$$

Some problems have more than one case. One way to solve such problems is to establish cases that cover all the possibilities. Calculate the number of arrangements for each case and then add the values for all cases to obtain the total number of arrangements.

MAT621B – Topic: Permutations, Combinations and the Binomial Theorem (C)

GCO: Develop algebraic and numeric reasoning that involves combinatorics.

GRADE 11 – MAT521B	GRADE 12 – MAT621B
	PC2 Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

SCO: **PC2 – Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).** **[CN, R, V]**

Students who have achieved this outcome should be able to:

A. Explain the patterns found in the expanded form of $(x+y)^n$, $n \leq 4$ and $n \in N$, by multiplying n factors of $(x+y)$.

B. Explain how to determine the subsequent row in Pascal's triangle, given any row.

C. Relate the coefficients of the terms in the expansion of $(x+y)^n$ to the $(n+1)$st row in Pascal's triangle.

D. Expand $(x+y)^n$ using the binomial theorem.

Section(s) in Pre-Calculus 12 *that address the specific curriculum outcome with relevant Achievement Indicators in brackets:*

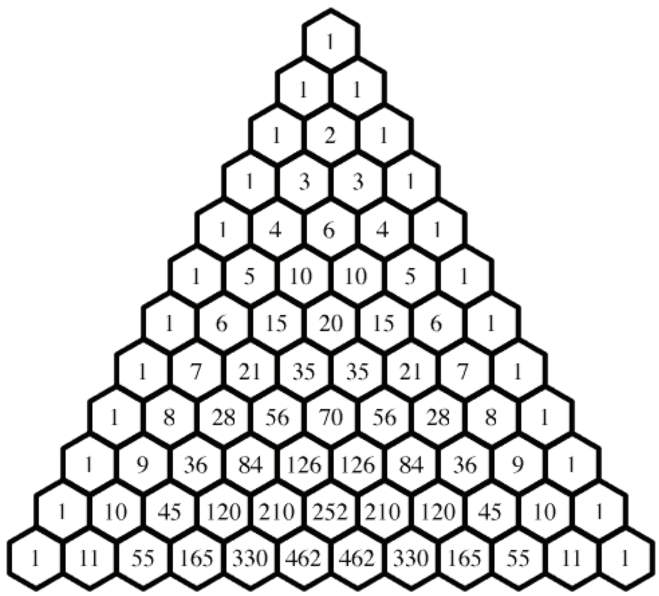
11.3 (A B C D)

[C]	Communication	**[ME]**	Mental Mathematics and Estimation	**[PS]**	Problem Solving	**[T]**	Technology
[CN]	Connections			**[R]**	Reasoning	**[V]**	Visualization

SCO: PC2 – Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). **[CN, R, V]**

Elaboration

Pascal's triangle has many patterns embedded within it. For example, each row begins and ends with 1. Each number in the interior of any row is the sum of the two numbers to its left and right in the row above. Each term in Pascal's triangle can be written as a combination of the form ${}_nC_r$, where n corresponds to the row number and r corresponds to the term number, beginning with Row 0 and Term 0.



It can be shown that the terms in Row $(n+1)$ of Pascal's triangle correspond to the coefficients in the expansion of $(x+y)^n$. This fact is used in the development of the binomial theorem, which can be used to expand any power of a binomial expression, $(x+y)^n$, where n is a natural number:

$$(x+y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$$

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